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PREFACE

This book is designed primarily for first-year students in colleges and technical schools. From teaching experience each of the authors feels that many students fail to grasp the essentials of trigonometry because too many new ideas are introduced at one time. The present book represents the results of an attempt to work out, in so far as possible, the cardinal principle of introducing only one important new idea at a time, and of offering guidance in teaching by emphasis on essentials.

In accord with this principle, we first define only the three principal trigonometric functions of an acute angle rather than the six functions of a general angle, as is often done. The meaning of the trigonometric functions is fixed permanently by their use in simple examples in Chapter I. As the next step, in Chapter II, numerous applications are given to heights and distances and to simple physical problems with theory restricted to that involved in the solution of right triangles without employing interpolation.

By a very natural learning process we then proceed to the general angle. Appropriate emphasis is placed on identities and the formal side of trigonometry in expressing functions of any angle in terms of the functions of acute angles.

A feature is introduced in Chapter V on the graphic representation of the functions by including questions to be answered from the graphs.

To proceed naturally without introducing more new ideas than necessary at one time, Chapter VI is devoted to the derivation of the *law of sines* and the *law of cosines* with many interesting applications. In this chapter the computations, made by the use of a four-place table of natural functions requiring interpolation, are relatively short due to the fact that only certain required parts of the oblique triangle are found.

The treatment of logarithms is almost entirely restricted to the base 10, and the best practice in calculations with logarithms is emphasized. Considerable attention is given to the question of the degree of accuracy to be reasonably expected in calculations by using tables of functions to a certain number of decimal places or to a certain number of significant figures.

The book contains at least twice as many exercises and problems as can be used in a standard course. While the easier exercises are carefully graded, a few really difficult exercises are given to stimulate the interest of the superior student. Answers are given in the back of the book to the odd-numbered exercises only. This meets the needs of those who prefer that students have a book without answers. An answer book giving answers to even numbered exercises is also available.

With respect to tables of functions and of logarithms, it has been our experience that first-year students tend to be confused by the presence of tables they do not use. On this account we have restricted our tables to those that are essential to a well rounded course in trigonometry.

We recognize that courses of various lengths are offered and we have prepared outlines for one course in which only thirty lessons are given and another in which forty-five lessons are given.

Some teachers who prefer not to follow a detailed schedule may find the following summary from the detailed schedules useful in apportioning time:

COURSE OF 30 LESSONS

Chapter	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Number of Lessons	5	3	4	1	1	3	3	3	5	1	1	0

COURSE OF 45 LESSONS

Chapter	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Number of Lessons	5	4	6	2	2	4	5	4	7	2	2	2

A longer course may well include spherical trigonometry.

The authors are indebted to their colleagues for many suggestions, and they are under a special debt of gratitude to Professor E. W. Chittenden for certain ideas and for valuable criticisms on the manuscript.

H. L. R.
J. F. R.
R. W.

Iowa City, Iowa
February, 1936

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CHAPTER I

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

1. On the Meaning and Scope of Trigonometry. The word *trigonometry* means literally the measurement of triangles. While trigonometry has many applications which depend on triangles, its subject matter consists largely of the theory of certain functions of angles called the *trigonometric functions.* These functions are of great mathematical importance, and have numerous applications to physics, engineering, and other fields.

2. Definition of the Three Principal Trigonometric Functions of an Acute Angle. Consider a right triangle ABC (Fig. 1). The acute angles at the vertices are denoted by the capital letters A, B, the right angle by C, and the sides opposite the angles by a, b, c, respectively (Fig. 1).

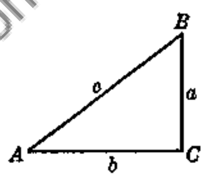


FIG. 1

With the three sides we may form the six ratios

$$\frac{a}{c},\quad \frac{b}{c},\quad \frac{a}{b},\quad \frac{b}{a},\quad \frac{c}{b},\quad \frac{c}{a}.$$

Since the values of these ratios (see § 4) are determined by the size of the angle A, they are called *trigonometric functions* of the angle A.

The first three of these ratios are called the principal trigonometric functions of the angle A. They are named and written as follows:

The ratio a/c is called the ***sine*** of A (written $\sin A$).
The ratio b/c is called the ***cosine*** of A (written $\cos A$).
The ratio a/b is called the ***tangent*** of A (written $\tan A$).

In describing the two legs of the right triangle ABC with reference to the angle A, side a (Fig. 1) is called the **opposite** side, and b the **adjacent** side. The above definitions of the three principal trigonometric ratios may then be stated as follows:

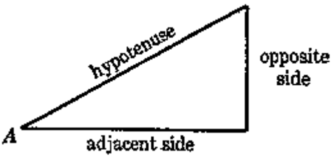


FIG. 2

$$\sin A = \frac{\textbf{opposite side}}{\textbf{hypotenuse}},$$

$$\cos A = \frac{\textbf{adjacent side}}{\textbf{hypotenuse}},$$

$$\tan A = \frac{\textbf{opposite side}}{\textbf{adjacent side}}.$$

EXERCISES

1. Given a right triangle with sides 3, 4, and 5 inches (Fig. 3) and with the side 4 inches opposite the angle A. Find the values of $\sin A$, $\cos A$, and $\tan A$.

2. Find the values of the sine, cosine, and tangent of each of the acute angles of a triangle with sides 5, 12, and 13. Draw a figure showing the triangle.

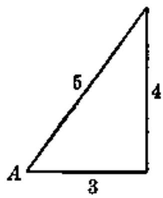


FIG. 3

3. The two legs of a right triangle are 7 and 24. Find the hypotenuse, and the values of the sine, cosine, and tangent of each of the acute angles. Draw a figure.

4. Construct a square of side 1 inch. Draw a diagonal. Its length is $\sqrt{2}$ inches. Why? From the figure, find the values of $\sin 45°$, $\cos 45°$, $\tan 45°$.

5. Construct an equilateral triangle of side 2 inches. Draw an altitude. Its length is $\sqrt{3}$ inches. Why? From the figure, find $\sin 60°$, $\cos 60°$, $\tan 60°$.

6. Given that the legs of a right triangle, opposite angles A and B, are 8 and 15 respectively. Find $\sin A$, $\sin B$, $\cos A$, $\cos B$, $\tan A$, and $\tan B$.

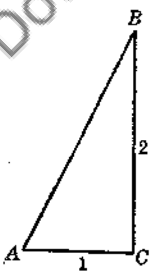


FIG. 4

7. The legs of a right triangle are 10 and $10\sqrt{3}$. Find the sine, cosine, and tangent of its smallest angle.

8. In a right triangle with one angle 20°, the hypotenuse is 100 feet, and the legs are 34.2 feet and 94.0 feet. Find $\sin 20°$, $\cos 20°$, $\tan 20°$.

9. Given a right triangle with base 1 foot and an angle at the base equal to 44°. The vertical side and hypotenuse are 0.97 feet and 1.39 feet respectively. Find the values of $\sin 44°$, $\cos 44°$, $\tan 44°$.

10. Given $\tan A = 2$. Construct the angle A with a ruler and compass.

SOLUTION. Let AC (Fig. 4) be of length 1. At C erect a perpendicular CB of length 2. Draw the line AB. The angle A is such that $\tan A = 2$.

11. Given $\sin A = 3/5$. Show how to construct the angle A with a ruler and compass.

SOLUTION. At a point C on a horizontal line AC (see Fig. 5), draw a vertical line CB of length 3. With B as a center and a radius 5, draw an arc cutting the horizontal line at some point A. Then angle A is such that $\sin A = 3/5$.

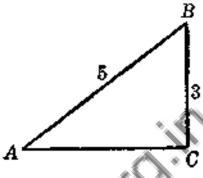


FIG. 5

12. Given $\cos A = 5/13$. Construct the angle A.

13. Given $\sin A = 1/2$. Construct the angle A.

14. Given $\tan B = 2$. Construct the angle B.

3. Cosecant, Secant, and Cotangent. In addition to the three principal trigonometric functions defined in § 2, considerable use is made of the three remaining ratios which are named and defined as follows:

$$\textit{cosecant of } A \text{ (written } \csc A) = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{1}{\sin A},$$

$$\textit{secant of } A \text{ (written } \sec A) = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{1}{\cos A},$$

$$\textit{cotangent of } A \text{ (written } \cot A) = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{1}{\tan A}.$$

It should be noted that the *cosecant, secant, and cotangent of an angle are, respectively, the reciprocals of the sine, cosine, and tangent of that angle.*

EXERCISES

1. What is meant by the reciprocal of a number? Given that

$$\csc A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \cot A = \frac{1}{\tan A},$$

show that

$$\sin A = \frac{1}{\csc A}, \quad \cos A = \frac{1}{\sec A}, \quad \tan A = \frac{1}{\cot A}.$$

2. Find the values of $\csc A$, $\sec A$, and $\cot A$ in a triangle with sides 3, 4, 5, and with the side 4 opposite the angle A.

3. The two legs of a right triangle are 9 and 40 with the side 9 opposite the angle A. Find the values of $\csc A$, $\sec A$, and $\cot A$.

4. The sides of a right triangle are 7, 24, and 25. Find the values of the cosecant, secant, and cotangent of each of its acute angles.

5. Given $\cot A = 4/3$. Construct the angle A with a ruler and compass.

6. Given $\sec A = 3$. Show how to construct the angle A with ruler and compass.

4. The Value of a Trigonometric Function Depends on the Angle only. The numerical value of a trigonometric function of an angle depends upon the size of the angle and not upon the size of the triangle to which the angle belongs. Thus, the numerical value of sin A (§ 2) does not change by changing the position of B on the line bounding the angle. This is seen to be true by observing that any two positions of B lead to similar triangles.

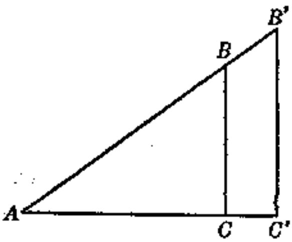


FIG. 6

Why are ABC and $AB'C'$ (Fig. 6) similar triangles? Are the ratios of corresponding sides equal in similar triangles? For example, in Fig. 6, is $CB/AB = C'B'/AB'$?

EXERCISES

1. In Fig. 6, can the value of sin A obtained from triangle ACB differ from that of sin A obtained from triangle $AC'B'$? Give reason for your answer.
2. Show from Fig. 6 that tan A obtained from the triangle ACB is equal to tan A obtained from triangle $AC'B'$.
3. When the leg opposite A is $5x$ and that adjacent to A is $12x$, find each of the six functions of A.

5. Functions of 45°. In Fig. 7, let angle $A = 45°$, then $B = 90° - 45° = 45°$. Hence, $a = b$. Why? Let $a = b = 1$. Then $c = \sqrt{1+1} = \sqrt{2}$. Why? From the definitions, § 2,

$$\sin 45° = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45° = 1,$$

$$\csc 45° = \sqrt{2},$$

$$\sec 45° = \sqrt{2},$$

$$\cot 45° = 1.$$

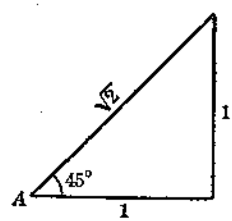


FIG. 7

6. Functions of 30° and 60°. In the equilateral triangle ABD (Fig. 8) let each side $= 2$. Draw CB perpendicular to AD, then AD is bisected at C, making $AC = 1$, and the angle $ABC = 30°$. Furthermore, $CB = \sqrt{2^2 - 1^2} = \sqrt{3}$. Hence

$\sin 30° = \frac{1}{2} = \cos 60°,$

$\cos 30° = \frac{1}{2}\sqrt{3} = \sin 60°,$

$\tan 30° = \frac{1}{3}\sqrt{3} = \cot 60°,$

$\csc 30° = 2 = \sec 60°,$

$\sec 30° = \frac{2}{3}\sqrt{3} = \csc 60°,$

$\cot 30° = \sqrt{3} = \tan 60°.$

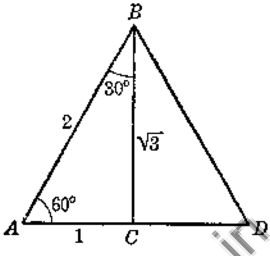


FIG. 8

7. Simple Examples whose Solutions Involve Trigonometric Functions of 30°, 45°, and 60°. The angles 30°, 45°, and 60° are used so frequently that it will be found convenient to be able to get the value of each of the six functions quickly from drawings such as those in Figs. 7 and 8.

EXAMPLE 1. When the sun is 30° above the horizon, how long a shadow is cast by a tree 100 feet high standing on level ground?

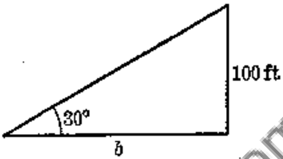


FIG. 9

SOLUTION. From Fig. 9, we have

$$\frac{b}{100} = \cot 30° = \sqrt{3}.$$

Hence $b = 100\sqrt{3}$* $= 173.21$ ft.

EXAMPLE 2. The top of a ladder AB (Fig. 10) which is 40 feet long rests on the upper edge of wall CB. The ladder has an inclination (angle A) of 45° with the horizontal. Find the height of the wall.

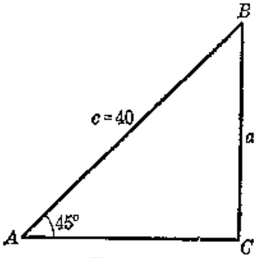


FIG. 10

SOLUTION. In Fig. 10, $\frac{a}{c} = \sin 45°.$ Then

$$a = c \sin 45° = 40\,\frac{\sqrt{2}}{2}^{*} = 28.28 \text{ ft.}$$

* It is desirable to memorize two square roots for solving problems in this book, namely,

$$\sqrt{2} = 1.4142, \quad \text{and} \quad \sqrt{3} = 1.7321,$$

which are approximations to four decimal places.

EXERCISES

1. A man climbs 1000 feet up a slope of 30° inclination. How high is he above the level of his starting point?

2. A kite is flying with 500 feet of stretched string out. If the string makes an angle of 45° with the horizontal, find the height of the kite.

3. If each of the equal sides of the right triangle used in finding the functions of 45° (§ 5) were a instead of 1, show that the values of the trigonometric ratios would remain as given in § 5.

4. Complete the following table with the exact values of sines, cosines, and tangents given in §§ 5 and 6.

angle	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	
45°			
60°			

5. By substituting approximate values of $\sqrt{2}$ and $\sqrt{3}$ in the table of Exercise 4, complete the following table giving approximate values to three decimal places.

angle	sin	cos	tan
30°	0.5	0.866	
45°			
60°			

6. The hypotenuse of a right triangle is 120 feet and one acute angle is 30°. Find the lengths of the other two sides.

7. When the sun is 60° above the horizon, how long a shadow is cast on level ground by a tree 80 feet high?

8. A monument standing on level ground is 75 feet high. How long a shadow is cast by this monument when the sun is 45° above the horizon?

9. How many degrees is the sun above the horizon when the length of the shadow of a pole is equal to the height of the pole? When the length of the shadow is equal to the height of the pole multiplied by $\sqrt{3}$?

Verify each of the following statements.

10. $\cos 60° = 2\cos^2 30° - 1$.

NOTE. $\cos^2 30°$ means $(\cos 30°)^2$. Similarly, $\sin^2 30°$ means $(\sin 30°)^2$.

11. $\cos 60° = 1 - 2\sin^2 30°$.

12. $\sin^2 30° + \sin^2 45° + \sin^2 60° = 3/2$.

13. $\sin 30° \cos 60° + \cos 30° \sin 60° = 1$.

14. $4\cot^2 45° - \sec^2 60° + \sin^3 30° = 1/8$.

Find the value of each of the following expressions and simplify it.

15. $6 \cos 60° + 8 \sin 30° - \tan 45°$
16. $8 \tan 30° - 8 \cot 60° + \cos 60° - 4 \cot 45°$
17. $\sin 30° \tan 45° \sec 60°$
18. $16 \cos 60° + 10 \cot 30° - 5\sqrt{3}$
19. $(11 \tan 60° - 4 \tan 45°)(11 \cot 30° + 4 \cot 45°)$
20. $(x \sin 60° - y \cos 30°) \csc 60°$
21. $\sin 30° + \cos 30° + \tan 30° + \csc 30° + \sec 30° + \cot 30°$

By substitution, verify each of the following statements.

22. $\sin 60° = 2 \sin 30° \cos 30°$.

23. $\cos 60° = \cos^2 30° - \sin^2 30°$.

24. $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$.

25. $\cos^2 30° = \dfrac{1 + \cos 60°}{2}$.

26. $\sin^2 30° = \dfrac{1 - \cos 60°}{2}$.

27. $2 \sin 30° + \cos 30° = \tan 45° + \sin 60°$.

28. $\tan 30° \sin 60° = \cos 60°$.

8. To Find the Values of Other Functions of an Acute Angle, when the Value of Any One Function is Given.

EXAMPLE 1. Given $\sin A = 3/5$, find the values of the other five functions.

SOLUTION. The equality $\sin A = 3/5$ tells us that in the right triangle we may in Fig. 11 take $CB = 3$ units, and $AB = 5$ units, whence $AC = \sqrt{5^2 - 3^2} = 4$ units. By definition, we then have

$$\cos A = \frac{4}{5} = 0.8, \qquad \sec A = \frac{5}{4} = 1.25,$$

$$\tan A = \frac{3}{4} = 0.75, \qquad \cot A = \frac{4}{3} = 1.3333$$

$$\csc A = \frac{5}{3} = 1.6667,$$

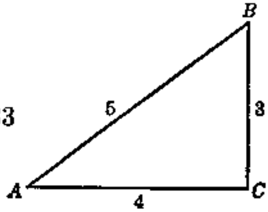


FIG. 11

EXAMPLE 2. Given $\sec A = 13/5$, find the values of the other five functions.

SOLUTION. The equality $\sec A = 13/5$ tells us that in the right triangle in Fig. 12, we may take $AC = 5$ and $AB = 13$, whence $CB = \sqrt{(13)^2 - 5^2} = 12$. We thus have

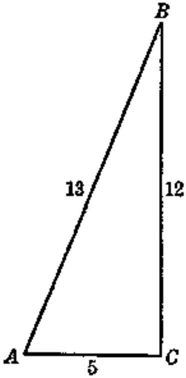


FIG. 12

$$\sin A = \frac{12}{13} = 0.9231,$$

$$\cos A = \frac{5}{13} = 0.3846,$$

$$\tan A = \frac{12}{5} = 2.4,$$

$$\csc A = \frac{13}{12} = 1.0833,$$

$$\cot A = \frac{5}{12} = 0.4167$$

EXAMPLE 3. Given $\tan A = 21/20 = 1.05$, find the values of the other five functions.

SOLUTION. The other functions are (Fig. 13)

$$\sin A = \frac{21}{29} = 0.7241,$$

$$\csc A = 1.3810, \quad \cos A = 0.6897$$
$$\sec A = 1.45, \quad \cot A = 0.9524$$

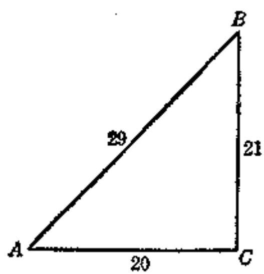


FIG. 13

EXAMPLE 4. To the six trigonometric functions defined above for an angle A may be added the ***versed sine*** of A (written **vers** A), ***coversed sine*** of A (written **covers** A), and ***haversine*** of A (written **hav** A). These are defined as follows:

$$\text{vers } A = 1 - \cos A,$$
$$\text{covers } A = 1 - \sin A,$$
$$\text{hav } A = \tfrac{1}{2}(1 - \cos A) = \tfrac{1}{2} \text{ vers } A.$$

Given $\sin A = 3/5$, find the values of vers A, covers A, and hav A.

SOLUTION. Substitute $\sin A = 3/5$ in covers $A = 1 - \sin A$, and we have

$$\text{covers } A = 1 - \frac{3}{5} = \frac{2}{5}.$$

From Example 1, $\cos A = 4/5$. Then

$$\text{vers } A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5},$$

and

$$\text{hav } A = \tfrac{1}{2} \text{ vers } A = \frac{1}{10}.$$

EXERCISES

Draw the angle A on coordinate paper, and obtain the value of each of the remaining trigonometric functions:

1. $\sin A = 4/5$.
2. $\csc A = 3$.
3. $\cot A = 24/7$.
4. $\cos A = 7/25$.
5. $\tan A = 5/12$.
6. $\sec A = 7$.
7. $\cos A = 15/17$.
8. $\sec A = 3$.
9. $\cot A = \sqrt{3}$.
10. $\sin A = \frac{1}{2}\sqrt{3}$.
11. $\cot A = \frac{1}{3}\sqrt{3}$.
12. $\tan A = \sqrt{3}$.
13. $\sin A = 0.5$
14. $\tan A = 2$.
15. $\csc A = 41/40$.
16. $\cos A = 0.6$
17. $\cot A = 0.75$
18. Given $\cos A = 7/25$, find vers A, covers A, and hav A.

Exercises 19–25 refer to a right triangle ABC (Fig. 14).

19. Given $\sin A = 3/5$ and $c = 20$. Find a.
20. Given $\tan A = 3/4$ and $b = 40$. Find a.
21. Given $\cos A = 2/3$ and $c = 24$. Find b.
22. Given $\csc A = 2.4$ and $a = 100$. Find c.
23. Given $\sec A = 4$ and $b = 50$. Find c.
24. Given $\cot A = 4/5$ and $a = 100$. Find b.
25. Given $\cos A = 4/5$ and $c = 100$. Find a and b.

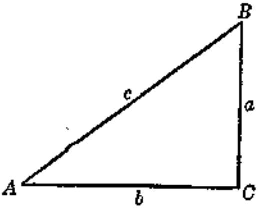


FIG. 14

9. Functions of Complementary Angles—Co-Functions. Two trigonometric functions, such as sine and cosine, in which the name of the one is obtained from that of the other by prefixing or dropping a first syllable "co" are called *co-functions*. In Fig. 15, the angles A and B are complementary, that is, $A + B = 90°$. By definition, § 2, we have

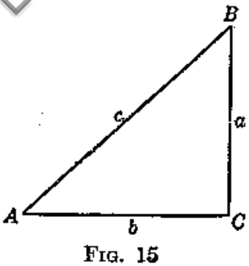


FIG. 15

$$\sin A = \frac{a}{c} = \cos B, \quad \csc A = \frac{c}{a} = \sec B,$$

$$\cos A = \frac{b}{c} = \sin B, \quad \sec A = \frac{c}{b} = \csc B,$$

$$\tan A = \frac{a}{b} = \cot B, \quad \cot A = \frac{b}{a} = \tan B.$$

That is, if two co-functions are equal, the sum of the angles is 90°. In another form, since $B = 90° - A$, we may write

$$\begin{aligned} \sin A &= \cos (90° - A), & \cos A &= \sin (90° - A), \\ \tan A &= \cot (90° - A), & \cot A &= \tan (90° - A), \\ \sec A &= \csc (90° - A), & \csc A &= \sec (90° - A). \end{aligned}$$

Hence *any trigonometric function of an acute angle A is equal to the co-function of the complement of A.*

EXERCISES

Fill in the blanks with the appropriate co-function:

1. $\sin 30°$ = ——.
2. $\cot 66°$ = ——.
3. $\tan (90° - 20°)$ = ——.
4. $\csc 67° 30'$ = ——.
5. $\sec 11° 25'$ = ——.
6. $\cot 27° 32'$ = ——.
7. $\cos 67° 30'$ = ——.
8. $\tan (90° - x)$ = ——.
9. Express the six functions of 75° as functions of an angle less than 45°.
10. Find the acute angle x if $\sin (x + 30°) = \cos x$.

SOLUTION. From § 9, the sum of the angles is 90°; that is,

$$\begin{aligned} x + 30° + x &= 90°, \\ 2x &= 60°, \\ x &= 30°. \end{aligned}$$

11. Find the acute angle x if $\sin (x + 30°) = \cos (30° + x)$.
12. Find the acute angle x if $\cos 2x = \sin (60° - x)$.
13. Find the acute angle x if $\tan 3x = \cot (60° - x)$.
14. Given the equation $\cot (x + 45°) = \tan (x + 15°)$, find the acute angle x.
15. Find the acute angle x if $\sec (x + 60°) = \csc x$.
16. In Fig. 16 give the sines and cosines of the acute angles marked, assuming that OMP, OSR, ORP, and RNP are right angles.

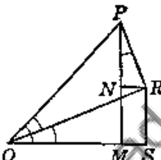


FIG. 16

10. Fundamental Relations among the Trigonometric Functions of an Acute Angle. For any acute angle A, we have the relation

$$\sin^2 A + \cos^2 A = 1. \tag{1}$$

That this relation holds may be seen from the right triangle, Fig. 17, in which the hypotenuse is 1, the side opposite A is $\sin A$, and that adjacent to A is $\cos A$.

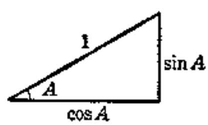


FIG. 17

Next, by definition (Fig. 17), we obtain the useful relations

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A},$$

(2)

$$\csc A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}.$$

Further, we shall need the relations

(3) $$\sec^2 A = 1 + \tan^2 A \quad \text{and} \quad \cot A = \frac{1}{\tan A}$$

which we may obtain from Fig. 18.

Finally, we shall need the relation

(4) $$\csc^2 A = 1 + \cot^2 A$$

which we may obtain from Fig. 19.

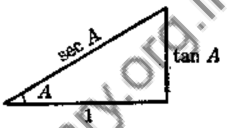


FIG. 18

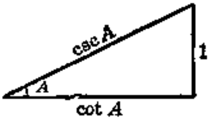


FIG. 19

The relations (1)–(4) are very useful for many purposes, and at this point we may see how they facilitate the simplification of certain algebraic expressions involving the trigonometric functions.

EXAMPLE 1. Simplify $\cos x \tan x + \sin x \cot x$.

SOLUTION. Replace $\tan x$ by its value $\frac{\sin x}{\cos x}$ from (2) and $\cot x$ by its value $\frac{\cos x}{\sin x}$. Then we have

$$\cos x \tan x + \sin x \cot x = \cos x \frac{\sin x}{\cos x} + \sin x \frac{\cos x}{\sin x}$$
$$= \sin x + \cos x.$$

EXAMPLE 2. Transform $\cot x + \frac{\sin x}{1 + \cos x}$ into $\csc x$.

SOLUTION. Replace $\cot x$ by $\frac{\cos x}{\sin x}$. Thus

$$\cot x + \frac{\sin x}{1 + \cos x} = \frac{\cos x}{\sin x} + \frac{\sin x}{1 + \cos x}$$
$$= \frac{\cos x + \cos^2 x + \sin^2 x}{\sin x(1 + \cos x)}$$
$$= \frac{\cos x + 1}{\sin x(1 + \cos x)} \quad \text{(Why?)}$$
$$= \frac{1}{\sin x} = \csc x.$$

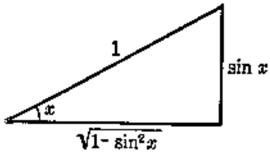


FIG. 20

EXAMPLE 3. Express the other trigonometric ratios in terms of $\sin x$.

Take a right triangle (Fig. 20) with 1 as a hypotenuse and $\sin x$ as the opposite side. Then

$$\text{base} = \sqrt{1 - \sin^2 x}, \qquad \csc x = \frac{1}{\sin x},$$

$$\cos x = \sqrt{1 - \sin^2 x}, \qquad \sec x = \frac{1}{\sqrt{1 - \sin^2 x}},$$

$$\tan x = \frac{\sin x}{\sqrt{1 - \sin^2 x}}, \qquad \cot x = \frac{\sqrt{1 - \sin^2 x}}{\sin x}.$$

EXERCISES

1. Simplify $1 - \cos^2 x$ by expressing it in terms of $\sin x$.
2. Simplify $(1 - \cos^2 x)(1 - \sin^2 x)$.
3. Simplify $(\sec^2 x - 1)(\csc^2 x - 1)$.
4. Transform $\dfrac{\cos x}{\sin x \cot^2 x}$ into $\tan x$.

Transform the first member of each of the following into the second member.

5. $(\tan x + \cot x) \sin x \cos x = 1$.
6. $\cos^2 x(1 + \tan^2 x) = 1$.
7. $\sin^2 x(1 + \cot^2 x) = 1$.
8. $\cos x + \tan x \sin x = \sec x$.
9. $\csc^2 x \tan^2 x - 1 = \tan^2 x$.
10. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$.
11. $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$.
12. $\sin x \cos x(\sec x + \csc x) = \sin x + \cos x$.
13. Express the other five trigonometric functions in terms of $\tan A$.

HINT. Use $\tan A$ as the opposite side and 1 as the adjacent side.

14. Express the other five functions in terms of $\cos A$.

MISCELLANEOUS QUESTIONS

By drawing figures, give a common sense reason rather than a formal proof for each of your answers to the following questions.

1. Can the sine of an angle be greater than 1?
2. Which is the larger, sin 10° or sin 20°?
3. Does the sine of an angle increase or decrease as the angle increases from 0° to 90°?
4. Which is the larger, cos 10° or cos 20°?

5. Does the cosine of an angle increase or decrease as the angle increases from 0° to 90°?

6. Which is the larger, tan 45° or tan 60°?

7. Does the tangent of an angle increase or decrease as the angle increases from 0° to 90°?

8. Can the cosine of an angle be greater than 1? Why?

9. Can the tangent of an angle be greater than 1? Is there any upper bound to the value of the tangent of an angle approaching 90°?

10. Does the cotangent of an angle increase or decrease as the angle increases from 0° to 90°?

CHAPTER II

SIMPLE APPLICATIONS: SOLUTION OF RIGHT TRIANGLES

11. Introduction. In the present chapter, we deal with simple applications of the trigonometric functions. In these applications the computations are most appropriately made by means of the values of the trigonometric functions of acute angles.

12. Three-Place Table of the Functions. In the computations of this chapter, we shall use the table of trigonometric functions found on pages 16–17. This table gives the values of the sine, cosine, tangent, and cotangent to three decimal places at intervals of $30'$ from $0°$ to $90°$. All angles will be read to the nearest $30'$. Thus, if $\sin A = 0.460$, the value selected for A is $27°\ 30'$, since $\sin 27° = 0.454$ and $\sin 27°\ 30' = 0.462$. If $\cot A = 0.902$, the value selected for A is $48°$. If $\cos A = 0.734$, we select for A the value $43°$ by agreeing to choose the nearest whole number of degrees when the value of the function lies midway between two tabulated values.

EXERCISES

Find from the table the value of each of the following functions.

1. $\sin 15°\ 30'$
2. $\sin 37°$
3. $\cos 22°\ 30'$
4. $\cos 33°\ 30'$
5. $\tan 37°$
6. $\tan 28°$
7. $\cot 9°\ 30'$
8. $\cot 43°$
9. $\sin 46°$
10. $\sin 77°$
11. $\cos 65°\ 30'$
12. $\cos 86°\ 30'$
13. $\tan 76°\ 30'$
14. $\tan 50°\ 30'$
15. $\cot 60°\ 30'$
16. $\cot 85°$

Find the acute angle A in each of the following equalities.

17. $\sin A = 0.242$
18. $\sin A = 0.725$
19. $\cos A = 0.588$
20. $\cos A = 0.623$
21. $\tan A = 0.424$
22. $\tan A = 1.511$
23. $\cot A = 1.054$
24. $\cot A = 0.167$

Find the acute angle A to the nearest 30 minutes in each of the following equalities.

25. $\sin A = 0.274$
26. $\sin A = 0.417$
27. $\cos A = 0.821$
28. $\cos A = 0.465$

29. $\tan A = 3.125$

30. $\tan A = 1.063$

31. $\cot A = 0.450$

32. $\cot A = 1.246$

In each of the following exercises, reduce the right-hand side to a decimal fraction and find the acute angle B, to the nearest 30 minutes.

33. $\sin B = \frac{16}{17}$.

34. $\sin B = 3/\sqrt{15}$.

35. $\cos B = \frac{213}{524}$.

36. $\cos B = \sqrt{7}/4$.

37. $\tan B = \frac{1}{7}\sqrt{8}$.

38. $\tan B = \frac{1}{11}$.

39. $\cot B = \frac{3.7}{0.13}$.

40. $\cot B = \frac{0.3}{\sqrt{0.12}}$.

13. Heights, Distances, and Angles. By the application of the definitions of the functions, and by the use of numerical values taken from the table (pp. 16–17), many interesting problems involving right triangles may be solved. This use of trigonometry will enable us to calculate distances and angles in situations in which it is impracticable to measure them.

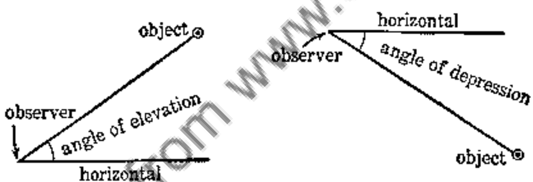


FIG. 21

DEFINITIONS. The ***angle of elevation*** of an object is the angle between the line of sight of the observer to the object and a horizontal line in the same vertical plane when the object observed is above the horizontal plane. When the object observed is below this horizontal plane, the angle is called the ***angle of depression*** (see Fig. 21).

EXAMPLE 1. A rope is stretched from the top of a telephone pole (Fig. 22) 36 feet high to a point on the level ground at which the angle of elevation of the top of the pole is 42°. How long is the rope if it is stretched taut?

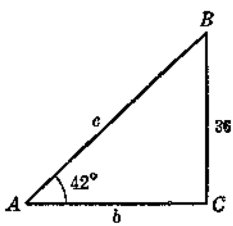


FIG. 22

SOLUTION. By definition, $36/c = \sin 42°$;

Three-Place Table of the Trigonometric Functions

A	sin A	cos A	tan A	cot A	A
0°	0.000	1.000	0.000	——	90°
0° 30′	0.009	1.000	0.009	114.6	89° 30′
1	0.017	1.000	0.017	57.29	89
1 30	0.026	1.000	0.026	38.19	88 30
2	0.035	0.999	0.035	28.64	88
2 30	0.044	0.999	0.044	22.90	87 30
3	0.052	0.999	0.052	19.08	87
3 30	0.061	0.998	0.061	16.35	86 30
4	0.070	0.998	0.070	14.30	86
4 30	0.078	0.997	0.079	12.71	85 30
5°	0.087	0.996	0.087	11.43	85°
5° 30′	0.096	0.995	0.096	10.39	84° 30′
6	0.105	0.995	0.105	9.514	84
6 30	0.113	0.994	0.114	8.777	83 30
7	0.122	0.993	0.123	8.144	83
7 30	0.131	0.991	0.132	7.596	82 30
8	0.139	0.990	0.141	7.115	82
8 30	0.148	0.989	0.149	6.691	81 30
9	0.156	0.988	0.158	6.314	81
9 30	0.165	0.986	0.167	5.976	80 30
10°	0.174	0.985	0.176	5.671	80°
10° 30′	0.182	0.983	0.185	5.396	79° 30′
11	0.191	0.982	0.194	5.145	79
11 30	0.199	0.980	0.203	4.915	78 30
12	0.208	0.978	0.213	4.705	78
12 30	0.216	0.976	0.222	4.511	77 30
13	0.225	0.974	0.231	4.331	77
13 30	0.233	0.972	0.240	4.165	76 30
14	0.242	0.970	0.249	4.011	76
14 30	0.250	0.968	0.259	3.867	75 30
15°	0.259	0.966	0.268	3.732	75°
15° 30′	0.267	0.964	0.277	3.606	74° 30′
16	0.276	0.961	0.287	3.487	74
16 30	0.284	0.959	0.296	3.376	73 30
17	0.292	0.956	0.306	3.271	73
17 30	0.301	0.954	0.315	3.172	72 30
18	0.309	0.951	0.325	3.078	72
18 30	0.317	0.948	0.335	2.989	71 30
19	0.326	0.946	0.344	2.904	71
19 30	0.334	0.943	0.354	2.824	70 30
20°	0.342	0.940	0.364	2.747	70°
20° 30′	0.350	0.937	0.374	2.675	69° 30′
21	0.358	0.934	0.384	2.605	69
21 30	0.367	0.930	0.394	2.539	68 30
22	0.375	0.927	0.404	2.475	68
22 30	0.383	0.924	0.414	2.414	67 30
A	cos A	sin A	cot A	tan A	A

Three-Place Table of the Trigonometric Functions

A	sin *A*	cos *A*	tan *A*	cot *A*	*A*
23°	0.391	0.921	0.424	2.356	67°
23° 30′	0.399	0.917	0.435	2.300	66° 30′
24	0.407	0.914	0.445	2.246	66
24 30	0.415	0.910	0.456	2.194	65 30
25	0.423	0.906	0.466	2.145	65
25 30	0.431	0.903	0.477	2.097	64 30
26	0.438	0.899	0.488	2.050	64
26 30	0.446	0.895	0.499	2.006	63 30
27	0.454	0.891	0.510	1.963	63
27 30	0.462	0.887	0.521	1.921	62 30
28°	0.469	0.883	0.532	1.881	62°
28° 30′	0.477	0.879	0.543	1.842	61° 30′
29	0.485	0.875	0.554	1.804	61
29 30	0.492	0.870	0.566	1.767	60 30
30	0.500	0.866	0.577	1.732	60
30 30	0.508	0.862	0.589	1.698	59 30
31	0.515	0.857	0.601	1.664	59
31 30	0.522	0.853	0.613	1.632	58 30
32	0.530	0.848	0.625	1.600	58
32 30	0.537	0.843	0.637	1.570	57 30
33°	0.545	0.839	0.649	1.540	57°
33° 30′	0.552	0.834	0.662	1.511	56° 30′
34	0.559	0.829	0.675	1.483	56
34 30	0.566	0.824	0.687	1.455	55 30
35	0.574	0.819	0.700	1.428	55
35 30	0.581	0.814	0.713	1.402	54 30
36	0.588	0.809	0.727	1.376	54
36 30	0.595	0.804	0.740	1.351	53 30
37	0.602	0.799	0.754	1.327	53
37 30	0.609	0.793	0.767	1.303	52 30
38°	0.616	0.788	0.781	1.280	52°
38° 30′	0.623	0.783	0.795	1.257	51° 30′
39	0.629	0.777	0.810	1.235	51
39 30	0.636	0.772	0.824	1.213	50 30
40	0.643	0.766	0.839	1.192	50
40 30	0.649	0.760	0.854	1.171	49 30
41	0.656	0.755	0.869	1.150	49
41 30	0.663	0.749	0.885	1.130	48 30
42	0.669	0.743	0.900	1.111	48
42 30	0.676	0.737	0.916	1.091	47 30
43°	0.682	0.731	0.933	1.072	47°
43° 30′	0.688	0.725	0.949	1.054	46° 30′
44	0.695	0.719	0.966	1.036	46
44 30	0.701	0.713	0.983	1.018	45 30
45	0.707	0.707	1.000	1.000	45
A	cos *A*	sin *A*	cot *A*	tan *A*	*A*

hence

$$c = \frac{36}{\sin 42°} = \frac{36}{0.669} = 53.8 \text{ ft.}$$

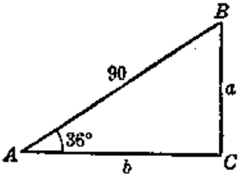


FIG. 23

EXAMPLE 2. The hypotenuse of a right triangle is 90 feet, and one acute angle is 36° (Fig. 23). Find the lengths of the other two sides.

SOLUTION. We have $a/90 = \sin 36°$, by definition; hence

$$a = 90 \sin 36° = (90)(0.588) = 52.9 \text{ ft.}$$

By definition, $b/90 = \cos 36°$; hence

$$b = 90 \cos 36° = (90)(0.809) = 72.8 \text{ ft.}$$

EXAMPLE 3. When the sun is 67° above the horizon, how long a shadow is cast by a tree 100 feet high standing on level ground (Fig. 24)?

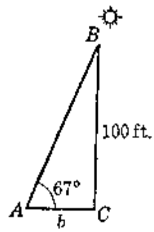


FIG. 24

SOLUTION. By definition, $b/100 = \cot 67°$; hence

$$b = 100 \cot 67° = (100)(0.424) = 42.4 \text{ ft.}$$

The answers to the following exercises are reported to three significant figures. For the meaning of significant figures and rules of approximate computation, see §§ 18, 19.

EXERCISES

1. A monument is 555 feet high. How long a shadow is cast by this monument when the sun is 62° above the horizon?

2. Gable rafters 22 feet long extend 2 feet beyond the outside of the walls of the house, and are set with an angle of elevation of 36°. Find the width of the house measuring from outside walls.

3. Find the radius of a circle in which a 10 foot chord subtends an angle of 18° at the center.

4. A straight trail having a uniform inclination of 17° leads from a hotel at an elevation of 9000 feet above sea level to a mountain inn at an elevation of 11 200 feet. Find the distance along the trail from the hotel to the mountain inn.

5. The length of a kite string is 225 yards. Find the height of the kite above the ground when its angle of elevation is 32°.

6. A man walks a mile up a path inclined 16° to the horizontal. How high is he above the level of his starting point?

7. From the top of a tower 325 feet high, the angle of depression of a rock on the horizontal plane through the foot of the tower is 35° 30′. Find the distance of the rock from the foot of the tower.

8. From the top of a monument 555 feet high, the angle of depression of the top of a flagpole 85 feet high is 56°. What is the horizontal distance from the monument to the flagpole?

9. A tree stands on level ground. At a point 132 feet from the tree and 12 feet above the ground, an observer found the angle of elevation of the top of the tree to be 32° 14′. Find the height of the tree.

10. What is the inclination of a roadbed having a grade of 8 per cent? One having a grade of 18 per cent? (A grade of 8 per cent means that the road has a rise of 8 feet in 100 feet on the horizontal.)

11. The angle of elevation of the top of a radio tower from a point at a horizontal distance of 1000 feet from its base is 37° 30′. Find the height of the tower.

12. A straight road runs down a long hill which is inclined to the horizontal at a constant angle of 17° 30′. The speedometer of an auto driven down the hill shows that the hill is 1.8 miles long. Through what vertical and horizontal distances has the auto travelled?

13. Find the angle of depression of a boat seen from a cliff 100 feet high if the horizontal distance of the boat from the cliff is 77 feet.

14. A guy wire of a silo is inclined to the ground at an angle of 37° 30′ and is 50 feet long. How high is the silo?

15. Find the angles of depression of the top and bottom of a flagpole, 84 feet high, from the top of a building 100 feet high, if the horizontal distance of the pole from building is 75 feet.

14. Applications in Physics and Geometry. Many problems occur in physics and geometry in which an angle is to be found from a formula that involves a trigonometric function of the angle. Thus, the horizontal and vertical components of a resultant velocity, v, in the path OP (Fig. 25) are given by

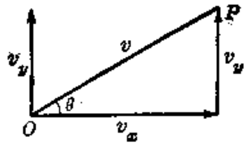


FIG. 25

$$v_x = v \cos\theta, \qquad v_y = v \sin\theta.$$

Identical relations hold if v, v_x, v_y are forces or accelerations instead of velocities.

EXAMPLE. A descending airplane has a horizontal velocity of 42 miles per hour and a resultant velocity of 50 miles per hour. At what angle with the horizontal is its path?

SOLUTION. In this case, $v = 50$ and $v_x = 42$. We are to find θ from the relation

$$v_x = v \cos \theta,$$

or

$$\cos \theta = \frac{v_x}{v} = \frac{42}{50} = 0.840.$$

Then, from the table, p. 17, $\theta = 33°$.

EXERCISES

1. A body is moving with a velocity of 34.2 feet per second in a path making an angle of 25° 30′ with the horizontal. Find the horizontal component.

2. A balloon is rising vertically at a rate of 552 feet per minute and due to the wind is drifting horizontally at a rate of 835 feet per minute. Find the angle made by the path of the balloon with the horizontal.

3. In symbols one of the laws of refraction is given by the relation

$$\frac{\sin i}{\sin r} = k,$$

where i is the angle of incidence, r is the angle of refraction, and k is the index of refraction. For rock salt $k = 1.54$, find r if $i = 31° 30'$.

4. Using the notation and the formula of Exercise 3,

(a) For water $k = 1.33$, find i if $r = 17° 30'$.
(b) For potassium chloride $k = 1.49$, find r if $i = 67°$.
(c) Find k if $i = 77° 30'$ and $r = 65°$.

5. The angle, ϕ, between two lines in a plane is given by the relation

$$\tan \phi = \frac{m_1{}^* - m_2}{1 + m_1 m_2}$$

where m_1 and m_2 are called the slopes of the lines. Find ϕ for the two lines whose slopes are 3 and $-1/7$ respectively.

6. In a space of three dimensions the angle, θ, between two lines is given by the relation

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are called the direction cosines of the lines. Find θ for the two lines whose direction cosines are 2/3, 1/3, 2/3 and 2/7, 6/7, 3/7, respectively.

7. One of the laws of friction is given by the formula

$$\tan i = k$$

where i is the inclination of the plane and k is the coefficient of friction. For iron and stone, k varies from 0.3 to 0.7. Between what angles does i vary?

* The symbol m_1 is read "m subscript one" or "m sub one."

8. To solve a problem in astronomy, it is necessary to make the following transformation

$$\tan N = \tan \delta \tan t.$$

If $t = 32°$ and $\delta = 37° 30'$, find N.

9. The angle ω is given by the formula

$$\cot \omega = \cot A + \cot B + \cot C.$$

Find ω if $A = 38° 30'$, $B = 67°$, $C = 74° 30'$.

15. Direction or Bearing of Lines and Points. To give the direction of a line in a horizontal plane near the surface of the earth, much use is made by surveyors and mariners of the acute angle which the line in question makes with the north-south line or meridian. This angle is often called the ***bearing of the line.***

The directions of all lines can be kept in terms of acute angles by reading from the north or south end of the meridian toward the east or west. Thus, in Fig. 26, the bearing of OA is N 40° E, that of OB is S 32° E, that of OC is S 70° W, that of OD is N 75° W.

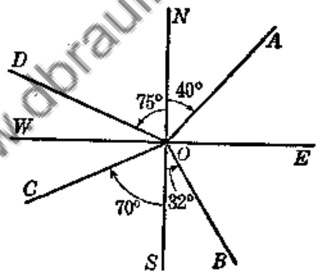


FIG. 26

It is sometimes convenient to speak of the ***bearing of a point*** with reference to another. For example, the bearing of the point A in Fig. 26 from the point O is N 40° E.

EXAMPLE. A ship leaves a dock at 8 o'clock and steams due east at a rate of 18 miles per hour. At 10 o'clock its course is changed to N 35° 30′ E. Find its distance and bearing from the dock at 11:30 o'clock.

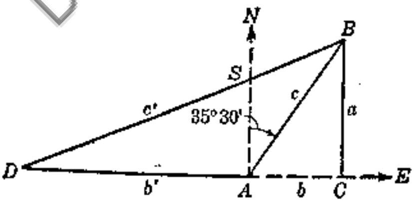


FIG. 27

SOLUTION. The values of known items are indicated in Fig. 27.

Let the ship start from the dock D, and alter its course at the point A, arriving at the point B at 11:30 o'clock. In the figure, $b' = 36$ miles, $c = 27$ miles. Since angle $SAB = 35° 30'$, it follows that

$$\text{angle } CAB = A = 54° 30'.$$

By definition, we have

$$\frac{b}{c} = \cos A, \qquad \text{or} \qquad b = c \cos A.$$

Substituting numerical values, we have

$$b = (27)(0.581) = 15.7 \text{ miles.}$$

Similarly, we find

$$a = c \sin A = (27)(0.814) = 22.0 \text{ miles.}$$

In the right triangle DCB, we have

$$DC = b' + b = 36 + 15.7 = 51.7 \text{ miles.}$$

Then

$$\tan D = \frac{a}{DC} = \frac{22}{51.7} = 0.426$$

From the table, p. 17, we have $\angle D = 23°$. Then the angle BSN is $90° - 23° = 67°$. Hence B bears N 67° E from D.

To calculate $DB = c'$, note that

$$\frac{a}{c'} = \sin D, \qquad \text{or} \qquad c' = \frac{a}{\sin D}.$$

Substituting, we have

$$c' = \frac{22}{0.391} = 56.3 \text{ miles.}$$

EXERCISES

1. Find the bearing of a point, P, 11 miles east and 14 miles north of a given point O.

2. Find the bearing of a straight railroad which leads to a point 20 miles east and 30 miles south.

3. What is the angle of elevation of the sun at the instant when the Washington monument, 555 feet high, casts a shadow 1000 feet long?

4. What is the angle of elevation of a 20-foot ladder which reaches 17 feet up a vertical wall?

5. Find the height of a tree which casts a horizontal shadow 40 feet long when the sun's elevation is 67° 30′?

6. One of the short sides of the rectangle 23 by 37 feet lies in a meridian; find the bearing of each vertex with respect to the opposite end of the diagonal through that vertex.

7. To find the width AB of a river, a surveyor measured a distance of 105 yards on a line AC at right angles to AB. From C he found the angle ACB to be 71° 30′. From the surveyor's data, calculate the required width of the river.

8. A flagpole stands on level ground. The angle of elevation of its top from a point on the ground 70 feet from its base is 59° 30′. What is the angle of elevation of its top from a point 200 feet from its base?

9. A surveyor, running a line due north, found that he must change his course because of a swamp. From a point A he alters his direction to N 55° E and chains 40 rods to a point C; thence on course N 33° W, he measures a distance 60 rods to a point B in his original course. What is the distance AB?

10. The angle of elevation of the top D of an inaccessible bluff from a point A is 30° 30′. At a point B, 200 feet nearer the base of the bluff, in a horizontal line with the point A, the angle of elevation of the top is 41°. How high is the bluff?

Suggestions. Let y be the height of the bluff. From Fig. 28, we have

$$y = x \tan 41°, \quad \text{and} \quad y = (x + 200) \tan 30°\,30'.$$

Eliminate x and solve for y.

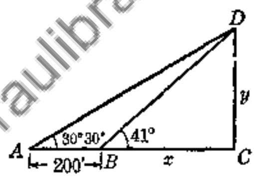


Fig. 28

11. A flagpole stands on level ground at the edge of a canal. At a point D on the opposite edge of the canal the angle of elevation of the top of the pole is 28°, but from a point A, 100 feet farther from the pole than D, in line with D and the pole, the angle of elevation of the top of the pole is 15° 30′. How wide is the canal?

Hint. Use suggestions in exercise 10.

12. From a given point in the horizontal plane of the base of the pedestal of the Statue of Liberty, the angles of elevation of the bottom and top of the statue, 151 feet high, are 4° 30′ and 9° 30′ respectively. Show that the height, y, of the pedestal is given by

$$y = \frac{151 \cot 9°\,30'}{\cot 4°\,30' - \cot 9°\,30'}$$

and find the approximate value of y.

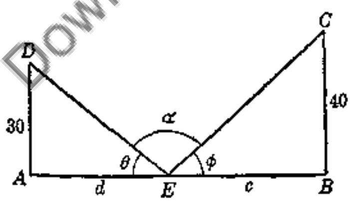


Fig. 29

In Fig. 29, $AD = 30$, $AB = 80$, and $BC = 40$. Use these data in the solution of the next four exercises.

13. Find $d = AE$ in Fig. 29 and the angle θ, if the angles of elevations of D and C from E are equal.

14. Find the distances AE such that the angle α (Fig. 29) is a right angle. Find the angles of elevation of D and C from that position of E which is nearer to A.

Hints. $\theta + \phi = 90°$, $\quad \dfrac{d}{30} = \cot \theta$, $\quad \dfrac{c}{40} = \cot \phi$.

But $\cot\theta = \tan\phi$. (See § 9.) Then $cd = 1200$, and $c + d = 80$. Solve the last two equations simultaneously.

15. A line is drawn from C (Fig. 29) perpendicular to a line BD (not shown in figure). It cuts AB in G. Find AG and the angle of elevation of C from G.

16. Find the vertical height of the intersection of BD and CG above AB.

17. Two observers, A and B, in a horizontal line passing through the base of a tower find that the angles of elevation of the top of the tower are θ and ϕ respectively. If A is farther from the tower than B, show that

$$x = h(\cot\theta \pm \cot\phi)$$

where x is the distance between A and B and h is the height of the tower.

18. In a horizontal line passing through the base of the Nashville, Tennessee, radio tower, 878 feet high, two observers find that the angles of elevation of the top of the tower are 55° and 40° 30′, respectively. How far apart are the observers?

19. From two consecutive milestones on a level road the angles of elevation of a balloon directly over the road are 37° and 47° 30′, respectively. Find the height of the balloon.

20. A crest of a hill is observed from the top and bottom of a building 100 feet high. The elevation angles are 31° and 35° 30′ respectively. Find the height of the crest of the hill above the bottom of the building.

21. From a point 20 feet above the surface of a lake the angle of elevation of the top of a church spire situated on the opposite shore is 11° 30′. The angle of depression of the tip of the reflection of the spire in the surface of the lake is 15°. Find the height of the spire above the surface of the lake and the horizontal distance of the spire from the point of observation.

HINT. Make use of the fact that the angle of incidence is equal to the angle of reflection.

22. Mount Washington is 6293 feet high. At a given point B in a horizontal line passing through its base, the angle of elevation of its summit is 53°. What is the angle of elevation of the summit at a point A in the same horizontal line as B at a distance one half mile farther from the base of the mountain than B?

16. Solution of Right Triangles. In the preceding articles we have seen how the right triangle is used in the solution of problems of a practical nature. A more formal treatment of the solution of a right triangle will now be given.

A triangle has six parts, or elements: three sides and three angles. A right triangle is determined by a side and any other part (not counting the right angle).

The primary formulas for computing the unknown parts of a right triangle are given by the definitions of the sine, cosine, tangent, and cotangent (see §§ 2 and 3). Thus the student will recall (Fig. 30) that

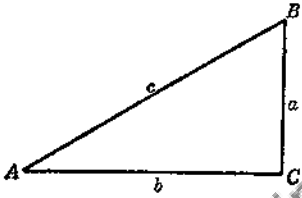


Fig. 30

(1) $\sin A = a/c = \cos B,$
(2) $\cos A = b/c = \sin B,$
(3) $\tan A = a/b = \cot B,$
(4) $\cot A = b/a = \tan B.$

To these are added the geometric relations

(5) $$A + B = 90°,$$
(6) $$a^2 + b^2 = c^2.$$

When one acute angle is given, the other may be found from (5). The *solution* of a right triangle is the process of finding the unknown parts from the given parts.

17. Procedure for Solving Right Triangles. The solution of a right triangle should be carried out in a systematic manner. In order to do this, note that each of the formulas in § 16, which contains a side, expresses a relation among three parts of a right triangle. Hence, from any formula which contains two given parts, the third part can be found.

Note also that it is advisable to select the formulas in such a manner that each part is computed from the given parts rather than from computed parts. This procedure will help to avoid the accumulation of errors.

The following numerical examples illustrate the procedure in the solution of right triangles.

Example 1. Given $A = 35° 30'$, $a = 21$, find B, b, and c.

Solution. The known parts are indicated in Fig. 31. From (5), § 16, $B = 90° - 35° 30' = 54° 30'$. Using (1) and (4), § 16, we have

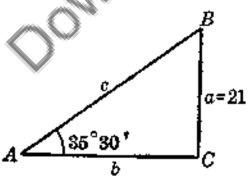


Fig. 31

$$\sin A = a/c, \quad \text{or} \quad c = a/\sin A,$$
$$\cot A = b/a, \quad \text{or} \quad b = a \cot A.$$

From the table, p. 17, $\sin A = 0.581$, and $\cot A = 1.402$

Upon substitution, we have

$$c = 21/0.581 = 36.1,$$
$$b = (21)(1.402) = 29.4$$

EXAMPLE 2. Given $b = 23$, $c = 36$, find a, A, and B.

SOLUTION. Draw a figure to represent the given data. From (6), § 16, we have

$$a^2 = c^2 - b^2 = (c - b)(c + b) = (36 - 23)(36 + 23)$$
$$= (13)(59) = 767.$$

Then $a = 27.7$

From (2), § 16, we have

$$\cos A = b/c = 23/36 = 0.639$$

From the table, p. 17, we have $A = 50° 30'$. Then $B = 39° 30'$.

EXERCISES

Solve each of the right triangles having given the following parts.

1. $a = 27$, $A = 31° 30'$.
2. $c = 29$, $A = 65°$.
3. $b = 24.5$, $A = 46° 30'$.
4. $a = 0.72$, $B = 15°$.
5. $a = 16$, $b = 18$.
6. $a = 31$, $c = 72$.
7. $b = 72$, $c = 100$.
8. $a = 76.8$, $A = 78°$.
9. $b = 80$, $A = 23° 30'$.
10. $c = 50$, $B = 48°$.

11. A rectangle is 10 by 30 feet. Find the length of the perpendicular dropped from a vertex upon a diagonal.

12. The vertical angle of an isosceles triangle is 67°. The base is 32 feet. Find the lengths of the altitudes.

13. In a circle of radius 10 feet, a chord is drawn 3 feet from the center. How large an angle does the chord subtend at the center of the circle?

14. At a point 17 feet from the center of a circle of radius 10 feet, tangents are drawn to the circle. Find the length of the tangents and the angle between them.

15. A traveller in a desert country finds that he must change his course. There are two routes open to him. First, he can turn right through an angle of 77 degrees and travel 38 miles; then turn left through a right angle and proceed to his destination. Second, he can turn left through an angle of 75 degrees and travel 44 miles; then turn right through a right angle and proceed to his destination. Which is the shorter route?

16. What is the radius of a circle inscribed in an equilateral triangle of perimeter 120 feet?

17. The angle between two 100 feet tangents to a circle is 71°. Find the radius of the circle.

18. A ladder whose foot is 30 feet from a building reaches 45 feet up the building. If the foot of the ladder is kept fixed, and the top is swung over against a building across an alley it reaches 50 feet up the building. What is the distance between the two buildings and the inclination of the ladder in its second position?

19. The drive wheel on a gas engine is 13 inches in diameter and the pulley on a rotary pump is 5 inches in diameter. If the shafts of the drive wheel and pulley are placed 2 feet apart, find the length of belt required when uncrossed. See Fig. 32. Use $\pi = 22/7$.

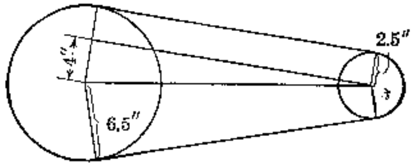


FIG. 32

18. Significant Figures. Decimal Places. Let us consider the meaning to be attached to the following statements:

(1) The temperature of a fever patient is 101.5°
(2) $\sqrt{2} = 1.414$
(3) A rainfall of 2.38 inches
(4) $\sin 18° = 0.309$

These statements mean that the exact values differ from the above approximations by at most 5 in the next decimal place. Thus, the temperature of the fever patient differs from 101.5° by at most 0.05°, that is, the exact temperature lies between 101.45° and 101.55° (the end values may at times be included). Similarly, $\sqrt{2}$ differs from 1.414 by at most 0.0005, the rainfall differs from 2.38 inches by at most 0.005 inch, and sin 18° differs from 0.309 by at most 0.0005.

The temperature, 101.5°, and the value, $\sqrt{2} = 1.414$, may each be said to be given to four significant figures; the rainfall, 2.38 inches, and the value $\sin 18° = 0.309$ to three significant figures.

Again, each of the values, $\sqrt{2} = 1.414$ and $\sin 18° = 0.309$, may be said to be given to three decimal places; the rainfall, 2.38 inches, to two decimal places, and the temperature, 101.5°, to one decimal place.

Consider the meaning of the statement that an estimate of 14 000 feet for the height of a mountain is correct to two significant figures. This means that the height of the mountain peak differs from 14 000 feet by at most 500 feet. But to say that a measurement of 14 000 feet for the height of the mountain peak is correct to five significant figures means that its height differs from 14 000 feet by at most 0.5 of a foot.

19. Reporting and Computing with Approximations. To report an approximate result such as 1.3462 to four significant figures, we give 1.346. To express the same result to three significant figures, we report 1.35 rather than 1.34 since 1.3462 is nearer to 1.35 than to 1.34. In like manner, to report a result such as 49.96 to three significant figures, we give 50.0. Note also that to express a result such as 2143.75 to only five significant figures, we report 2143.8 by agreeing to adopt the practice of reporting the nearest even number when the number whose approximate value we want lies midway between two numbers.

In multiplying two numbers of which at least one is an approximation, nothing is ordinarily gained in retaining more significant figures than occur in the approximate factor that contains the smallest number of significant figures. Thus, we may write $(12.1)(0.86) = 10$ if 0.86 is an approximate value, but we should write $(12.1)(0.86) = 10.4$ if 0.86 is an exact value and 12.1 is approximate.

A similar procedure applies also to division. For example, we may write $6.812/2.21 = 3.08$ if 2.21 is an approximate value, but we should write $6.812/2.21 = 3.082$ if 2.21 is an exact value and 6.812 is an approximate value.

In addition and subtraction of two numbers of which at least one is approximate, it is seldom useful to retain more decimal places than are in the number containing the least number of decimal places. Thus, if we add the approximate values:

$$\begin{array}{r} 1.356 \\ 14.5 \\ 0.26 \\ \hline 16.1 \end{array}$$

the result should be written in the form 16.1, as stated above.

In adding approximate values, errors may accumulate. Thus, in adding the following approximate values

$$\begin{array}{r} 36.52 \\ 27.31 \\ 26.24 \\ 12.32 \\ \hline \end{array}$$

in which an error as large as 0.005 may possibly exist in each of the four numbers, a maximum error of numerical value 4(0.005) = 0.02 might occur in the result. In general the error is relatively small in comparison with the maximum.

The observance of the above rules about reporting results in computing with approximations produces, in general, good results, but no attempt is made in this book to justify this statement.

In a practical problem, the measured quantities are, in general, approximate. In order to avoid confusion, we shall consider all data in this book exact, unless stated to the contrary. Since the values of the trigonometric functions in the table are approximate except in relatively few cases, the results computed by means of them are approximate. In the present chapter, approximate results have been reported to three significant figures, and have been obtained by following the practice of retaining the appropriate number of significant figures at the end of each step. However, some differences in the approximate results may occur due to the *order* in which the operations are performed.

EXERCISES

1. Given 78.29613, 0.01384, 1.11246, 0.00072198, as a set of approximate results, report them to three decimal places.

2. Write each of the numbers in Exercise 1 to three significant figures.

3. Write each of the numbers in Exercise 1 to four decimal places, to four significant figures.

4. Multiply (76.1)(8.6)(1.240) in order from left to right retaining the appropriate number of significant figures in each product, (1) when all the factors are approximate, (2) when the first two factors are exact and the third approximate, and (3) when the middle factor is exact and others approximate.

5. Do Exercise 4 using the factors in the order (1.240) (76.1) (8.6).

6. Perform the indicated operations (12.35)(3.34) ÷ 2.7 in order from left to right retaining the appropriate number of significant figures at each step, (1) when all the numbers are approximate, (2) when the divisor is exact and the others approximate, and (3) when the number 12.35 is approximate and the others exact.

7. Work Exercise 6 following the order [(12.35)/(2.7)] (3.34).

For the angles indicated in each of Exercises 8–10, test whether the equality is satisfied by values of the functions taken from the table, pp. 16–17.

8. $\sin^2 A + \cos^2 A = 1$, for $A = 34°$.

9. $\sin 2A = 2 \sin A \cos A$, for $A = 25° 30'$.

10. $\sin (A - B) = \sin A \cos B - \cos A \sin B$, for $A = 65° 30'$, and $B = 35°$.

Find the value of each of the expressions in Exercises 11–14, using the table on pp. 16–17 and assuming all numbers exact except the values of the trigonometric functions.

11. $\dfrac{(67.8)(\sin 28°)}{\sin 56° 30'}$

12. $\dfrac{(12.3)(\tan 21°)}{\cot 14° - \cot 23°}$

13. $\dfrac{(2.45)(18.7)}{\tan 56° - \tan 39° 30'}$

14. $\sin A + (1.3) \cos \left(\dfrac{A}{2}\right) - (13) \tan \left(\dfrac{A}{3}\right)$, if $A = 78°$

15. Give the results of the following multiplications to the appropriate number of significant figures: 44 sin 19°; 44.4 sin 1°; 4444 cos 1°.

16. Find the acute angle A, if $\tan (A - 17°) = 856/767$.

17. Find the acute angle A, if $\sin (A + 15°) = 2367/4321$.

CHAPTER III

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

20. The Extended Notion of an Angle. Coordinate Axes. In §§ 2–3, the trigonometric functions of positive acute angles are defined.

The meaning of an angle as used in elementary geometry will now be extended to include both positive and negative angles of any size, and the trigonometric functions of such angles will then be defined.

To do this we introduce ***coordinate axes.*** In Fig. 33, the line $X'X$ is called the x-axis, and the line $Y'Y$, perpendicular to $X'X$, is called the y-axis. The point of intersection, O, of $X'X$ and $Y'Y$ is called the ***origin.***

The coordinate axes divide the plane into four quadrants, I, II, III, IV, numbered as in Fig. 33.

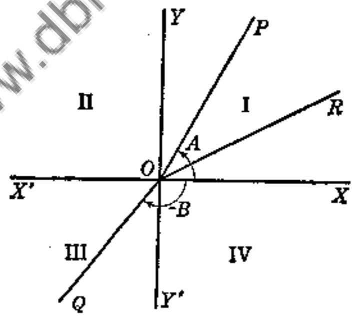


FIG. 33

21. Positive and Negative Angles. If a line, say OR in Fig. 33, rotates in a plane about one of its points O an angle is generated. When the rotation is counter clockwise we speak of the angle generated as a *positive* angle, but when the rotation is clockwise as a *negative* angle.

22. Initial and Terminal Sides of an Angle. If the line OR rotates from OX to OP, Fig. 33, we call OX the ***initial*** side and OP the ***terminal*** side of the positive angle A. Similarly OX and OQ are the initial and terminal sides of the angle $-B$.

23. The Quadrant of an Angle. Coterminal Angles. An angle may be so placed that its initial side will coincide with OX, Fig. 33, and its vertex with the origin. It is then said to be in ***standard*** position.

When placed in standard position an angle is said to belong to the quadrant in which the terminal side lies. Thus, in Fig. 33, the positive angle A is in standard position, and is a first quadrant angle. The angle $-B$ is in standard position, and is a third quadrant angle.

Angles in standard position whose terminal sides coincide may differ by multiples of 360°. Thus, the angles 40°, 400°, − 320°, ···, when in standard position, have the same terminal side. Such angles are called ***coterminal*** angles.

When speaking of coterminal angles, we will always assume that they are in standard position.

NOTE. The student should bear in mind that when we speak of an angle in a certain quadrant, say a second quadrant angle, we do not mean that the angle lies entirely in the second quadrant. We simply mean that the terminal side lies in the second quadrant.

EXERCISES

1. State the quadrant to which each of the following angles belongs: 120°, 375°, 210°, 510°, − 100°, − 30°, − 495°, − 320°.

2. State the quadrant to which each of the following angles belongs: $2 \cdot 90° + 30°$, $3 \cdot 90° - 40°$, $5 \cdot 90° + 20°$, $-4 \cdot 90° + 75°$, $-5 \cdot 90° - 30°$, 1300°, − 1765°.

3. Which pairs of the following angles are coterminal: 30°, 210°, 390°, − 330°?

4. Which pairs of the following angles are coterminal: − 60°, 300°, − 420°, 270°?

5. Give two positive angles and two negative angles having the same terminal side as 79° in standard position.

6. Give two positive angles and two negative angles having the same terminal side as − 117° in standard position.

7. What angle does the minute hand of a watch generate from 11 A.M. to 2 P.M. the same day?

8. What angle does the hour hand of a clock generate in $12\frac{1}{2}$ hours?

9. In what quadrant is the supplement of 70°?

10. In what quadrant is the complement of − 25°?

11. In a circle whose circumference is 200 feet, a central angle subtends an arc of 180 feet. To what quadrant does this angle belong?

12. Express an angular velocity of 156 complete revolutions per minute in degrees per second.

13. A wheel is revolving with an angular velocity of 900° per second. Find the number of complete revolutions per hour.

24. Rectangular Coordinates. The point P in any position, Fig. 34, is determined by two numbers, x and y. The distance, x, of the point P from the y-axis is called the ***abscissa*** of P. The distance, y, of the point P from the x-axis is called the ***ordinate***

of P. The abscissa, x, and the ordinate, y, of any point are called its rectangular coordinates, and are written (x, y).

The abscissa, x, of the point P is said to be positive if P is to the right of the y-axis, and negative if P is to the left of the y-axis. Similarly, the ordinate, y, is said to be positive if P is above the x-axis, and negative if P is below the x-axis.

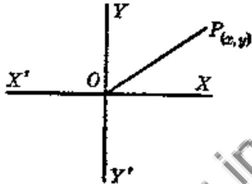


FIG. 34

The abscissa is zero if P is on the y-axis, and the ordinate is zero if P is on the x-axis.

The distance, r, of P from the origin, O, is called the ***distance*** or ***radius vector*** of the point P. In the definitions of the trigonometric functions (§ 25), we agree to take the distance, r, positive and different from zero.

EXERCISES

1. Locate the points $(4, -2)$, $(-3, 5)$, $(2, -3)$, $(-5, -4)$.
2. In which quadrant is each point in Exercise 1?
3. Find the distance, r, of each point in Exercise 1 from the origin.
4. Locate the points $(5, 12)$ and $(-5, 12)$.
5. How far is the point $(-24, -7)$ from the origin?

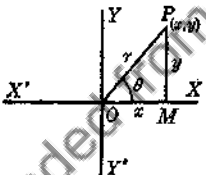


FIG. 35

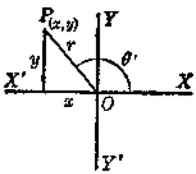


FIG. 36

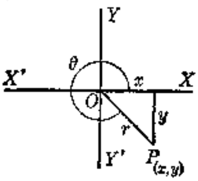


FIG. 37

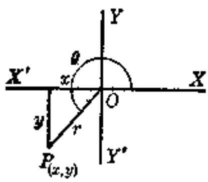


FIG. 38

25. Definitions of the Trigonometric Functions of Any Angle. Consider the right triangle OMP, in Fig. 35. Denoting the positive acute angle XOP by θ, we have, §§ 2–3,

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate of } P}{\text{distance of } P}, \qquad \csc\theta = \frac{r}{y} = \frac{\text{distance of } P}{\text{ordinate of } P},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa of } P}{\text{distance of } P}, \qquad \sec\theta = \frac{r}{x} = \frac{\text{distance of } P}{\text{abscissa of } P},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate of } P}{\text{abscissa of } P}, \qquad \cot\theta = \frac{x}{y} = \frac{\text{abscissa of } P}{\text{ordinate of } P}.$$

Now consider each of the four figures, Figs. 35–38. If θ denotes either the positive or the negative angle XOP in each figure, we take the above to be the definitions of the trigonometric functions of the angle θ in whatever quadrant the terminal side of the angle lies.

Consider, for example, the angle $\theta = XOP$ in Fig. 36. Let $P(-3, 4)$ be a point on the terminal side of the angle θ. We then have $r = 5$, and

$$\sin\theta = \frac{4}{5}, \qquad \cos\theta = -\frac{3}{5}, \qquad \tan\theta = -\frac{4}{3}.$$

EXERCISES

1. If θ is an angle in the third quadrant for which $x = -5$, and $y = -12$, find the values of $\sin\theta$, $\tan\theta$, and $\sec\theta$.

2. If θ is an angle in the fourth quadrant for which $x = 3$, and $y = -4$, find the values of $\sin\theta$, $\cos\theta$, and $\cot\theta$.

3. If θ is a second quadrant angle for which $r = 13$, $x = -12$, find $\cos\theta$, $\cot\theta$, and $\csc\theta$.

4. If θ is a fourth quadrant angle for which $r = 10$, $y = -8$, find $\sin\theta$, $\tan\theta$, and $\sec\theta$.

5. If θ is a second quadrant angle for which $x = -2$, $y = b$, find $\cos\theta$, $\tan\theta$, and $\csc\theta$.

6. The 7–24–25* right triangle may be so placed that its hypotenuse is the terminal side of two angles belonging to the second quadrant; find their sines, cosines, and tangents.

7. The 3–4–5 right triangle may be so placed that its hypotenuse is the terminal side of two angles belonging to the third quadrant; find their cosines, cotangents, and cosecants.

8. The 5–12–13 right triangle may be so placed that its hypotenuse is the terminal side of two angles belonging to the fourth quadrant; find their sines, tangents, and secants.

NOTE. It may be noted that in certain cases the definitions of the trigonometric functions fail due to a zero denominator. For example, the definition $\tan\theta = y/x$ fails when $x = 0$. This will be discussed in §§ 27 and 28.

* By a 7–24–25 right triangle we mean a right triangle with legs 7 and 24, and with hypotenuse 25.

26. Signs of the Trigonometric Functions. The signs of the trigonometric functions of an angle depend upon the quadrant to which the angle belongs. Visualize the terminal side of an angle in any quadrant. Note the signs of the abscissa and ordinate of any point on this terminal side, bearing in mind that the distance of the point from the origin is always taken as positive. Then the signs of the functions follow readily from the definitions.

For example, $\tan\theta = y/x$ is positive when x and y have like signs, and negative when they have unlike signs. But x and y have like signs for points in the first and third quadrants, and unlike signs for points in the other two quadrants. Hence $\tan\theta$ is positive when θ is a first or third quadrant angle, and negative when a second or fourth quadrant angle.

The signs of the trigonometric functions in the respective quadrants are given in Fig. 39.

II In this quadrant the *sine* and *cosecant* only are positive	I In this quadrant *all* the functions are positive
III In this quadrant the *tangent* and *cotangent* only are positive	IV In this quadrant the *cosine* and *secant* only are positive

FIG. 39

EXERCISES

1. Why are the signs of $\sin\theta$ and $\csc\theta$ in any quadrant the same as the signs of the ordinates of points in that quadrant?

2. Why are the signs of $\cos\theta$ and $\sec\theta$ in any quadrant the same as the signs of the abscissas of points in that quadrant?

3. Draw a figure and find each of the six trigonometric functions of 135°.

SOLUTION. Let P (Fig. 40) be the point $(-1, 1)$. Then the line OP bisects the angle YOX', and $\angle XOP = 135°$.

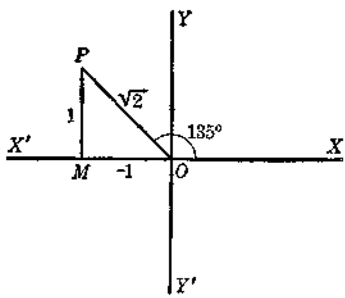


FIG. 40

By the definitions, we have

$$\sin 135° = \frac{1}{\sqrt{2}}, \qquad \csc 135° = \sqrt{2},$$

$$\cos 135° = -\frac{1}{\sqrt{2}}, \qquad \sec 135° = -\sqrt{2},$$

$$\tan 135° = -1, \qquad \cot 135° = -1.$$

If P were the point $(-3, 3)$, would the line OP bisect the angle YOX'?

4. By using Fig. 40, find each of the trigonometric functions of $-225°$.

5. Draw a figure and find each of the trigonometric functions of 225°.

HINT. Note that the point $(-1, -1)$ lies on the terminal side of 225°.

6. Use the figure drawn in Exercise 5 to find the six functions of $-135°$.

7. From a figure find each of the trigonometric functions of 315°; $-45°$.

8. Complete the following table from the results of Exercises 3 to 7.

angle	sin	cos	tan	csc	sec	cot
45°, −315°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
135°, −225°						
225°, −135°						
315°, −45°						

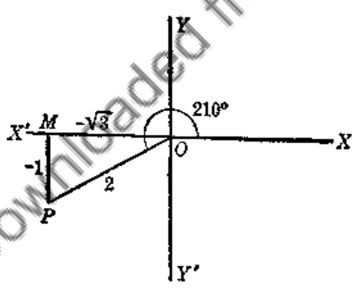


FIG. 41

9. From Fig. 41, find each of the trigonometric functions of 210°, $-150°$.

10. Draw a figure and find the six trigonometric functions of 240°, $-120°$.

11. Draw a figure and find the six trigonometric functions of 300°, 330°, $-60°$, $-30°$.

12. Draw a figure and find the six trigonometric functions of 120°, 150°, $-240°$, $-210°$.

Evaluate the following:

13. $4 \sin 120° - \cot 150°$

14. $\sin 30° + \sin 150° + \cot 60° + \cot 120°$

15. $(\cos 300° + \tan 135°)(\cos 300° - \tan 135°)$

16. $(\cos 240° \sin 330°) + (\cos 120° \sin 150°)$

17. $\cot 135° + \sec 240° + \sin^2 300°$

18. $\sec^2 150° - \tan^2 120°$

19. Given $\tan \theta = 5/12$, find the values of the other functions of θ.

SOLUTION. The angle θ may be either the positive or the negative angle XOP, Fig. 42, or the positive or the negative angle XOQ, Fig. 43.

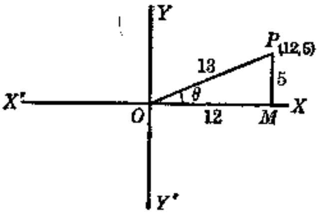


FIG. 42

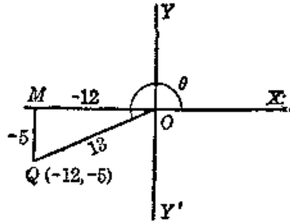


FIG. 43

In Fig. 42, $\tan\theta = 5/12$; in Fig. 43, $\tan\theta = (-5)/(-12) = 5/12$. The distance $OP = \sqrt{5^2 + 12^2} = 13$; the distance $OQ = \sqrt{(-5)^2 + (-12)^2} = 13$. The values of the other functions are given in the following table:

angle	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\theta = XOP$	$\frac{5}{13}$	$\frac{12}{13}$		$\frac{13}{5}$	$\frac{13}{12}$	$\frac{12}{5}$
$\theta = XOQ$	$-\frac{5}{13}$	$-\frac{12}{13}$		$-\frac{13}{5}$	$-\frac{13}{12}$	$\frac{12}{5}$

Draw the angles θ, and find the values of the other functions, expressing results with rational denominators, given:

20. $\sin\theta = -15/17$.

21. $\cos\theta = 2/7$.

22. $\tan\theta = -5/12$.

23. $\csc\theta = 3$.

24. $\sec\theta = -2$.

25. $\cot\theta = 2\sqrt{6}$.

Draw the angle θ in the appropriate quadrants, and find the other functions, given:

26. $\cot\theta = 2x/3$, where x is positive.

27. $\cos\theta = m$, where m is negative.

28. $\csc\theta = (1 + m^2)/(2m)$, where $0 < m < 1$.

29. $\sin\theta = l/m$, a positive fraction.

30. $\cos\theta = l/m$, a positive fraction.

In Exercises 31 and 32, the fraction l/m can always be written with the denominator positive.

31. $\sin\theta = l/m$, a negative fraction.

32. $\cos\theta = l/m$, a negative fraction.

33. $\tan\theta = l/m$, a positive fraction.

HINT. In Exercise 33, l and m have like signs.

34. $\tan\theta = l/m$, a negative fraction.

HINT. In Exercise 34, l and m have unlike signs.

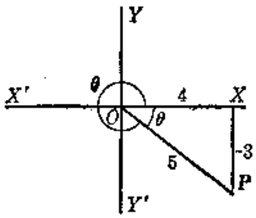


FIG. 44

35. Given $\cos\theta = 4/5$, and $\tan\theta$ negative, find the values of the other functions.

SOLUTION. Since $\cos\theta$ is positive, the angle θ must be in either the first or fourth quadrant. Since $\tan\theta$ is negative the angle θ must be in either the second or fourth quadrant. Hence the angle θ can be in the fourth quadrant only (Fig. 44). The values of the other functions are given in the following table.

angle	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\theta = XOP$	$-\frac{3}{5}$		$-\frac{3}{4}$	$-\frac{5}{3}$	$\frac{5}{4}$	$-\frac{4}{3}$

Draw the angle θ and find the values of the other functions, given:

36. $\cos\theta = -12/13$, and $\tan\theta$ positive.
37. $\csc\theta = -2$, and $\cos\theta$ negative.
38. $\cot\theta = -\sqrt{7}/3$, and $\sin\theta$ positive.
39. $\sin\theta = 0.6$, and $90° < \theta < 180°$.
40. Given $\cot\theta = \sqrt{5}/2$, and $\sec\theta < 0$, find the value of

$$\frac{\cos\theta + \csc\theta}{\sin\theta + \sec\theta}.$$

27. Quadrantal Angles. When the terminal side OP of an angle in standard position falls on the boundary between two quadrants, the angle is called a ***quadrantal angle***. The angles, 0°, 90°, or any multiple of 90°, are quadrantal angles.

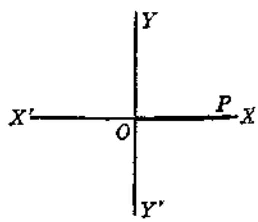


FIG. 45

Consider the quadrantal angle, 0°. The terminal side OP of the angle 0° falls on the boundary between the first quadrant and the fourth quadrant (Fig. 45). For this angle, $x = r$ and $y = 0$. By definition (§ 25), we have

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0, \qquad \csc 0° = \frac{r}{y} = \frac{r}{0}^*,$$

$$\cos 0° = \frac{x}{r} = \frac{r}{r} = 1, \qquad \sec 0° = \frac{r}{x} = \frac{r}{r} = 1,$$

$$\tan 0° = \frac{y}{x} = \frac{0}{r} = 0, \qquad \cot 0° = \frac{x}{y} = \frac{r}{0}^*.$$

* Since division by zero is an impossible operation, the definitions of § 25 fail for csc 0° and cot 0°. See § 28.

For the angle 90° the terminal side falls on the boundary between the first and second quadrants (Fig. 46). For this angle, $x = 0$ and $y = r$.

By applying the definitions in § 25 to the angle 90°, we obtain the following values of the functions:

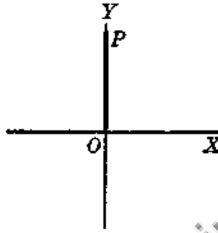


FIG. 46

$$\sin 90° = \frac{y}{r} = \frac{r}{r} = 1, \qquad \csc 90° = \frac{r}{y} = \frac{r}{r} = 1,$$

$$\cos 90° = \frac{x}{r} = \frac{0}{r} = 0, \qquad \sec 90° = \frac{r}{x} = \frac{r}{0}^*,$$

$$\tan 90° = \frac{y}{x} = \frac{r}{0}^*, \qquad \cot 90° = \frac{x}{y} = \frac{0}{r} = 0.$$

Here the definitions fail for tan 90° and sec 90°.

The student should draw a figure and write out the values of the functions of 180°, 270°. It should be noted that the definitions fail for csc 180°, cot 180°, tan 270°, and sec 270°.

28. Cases in Which the Definitions Fail. In the eight cases noted in § 27 in which the definitions of § 25 fail it is customary to say that the functions are not *defined*. That these functions have *some meaning*, however, may be seen as follows:

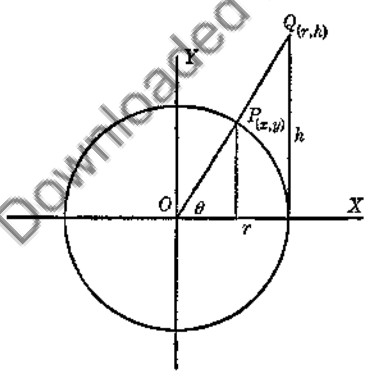


FIG. 47

Take, for example, tan 90°. It is not defined, that is, the angle 90° has no tangent. But every other angle in the neighborhood of 90° has a tangent equal to $y/x = h/r$, Fig. 47. As the angle θ approaches 90°, the denominator r remains fixed while the numerator h increases without bound. Hence $\tan \theta$ increases without bound when the angle θ approaches 90°. To express this fact briefly, we say that $\tan \theta$ becomes infinite when $\theta = 90°$. The customary abbreviated statement $\tan 90° = \infty$, read "tan 90° is infinite," does not mean that a value is assigned to tan 90°.

A similar discussion will show that each of the other seven functions csc 0°, cot 0°, sec 90°, csc 180°, cot 180°, tan 270°, and sec 270° is infinite.

EXERCISES

1. Fill the blanks in the following table:

angle	sin	cos	tan	csc	sec	cot
0°	0	1	0	∞	1	∞
90°						
180°		-1				
270°						

Find the values of the following:

2. $3 \cos 0° + 2 \sin 90° + \tan 180°$
3. $\cos 270° + \tan 180° + \sin 0°$
4. $\sin 360° + \csc 90° - 2 \sec 180°$
5. $\sin 0° \cos 90° + \tan 180° \csc 270°$
6. $\cos 180° + \cos 90° - \sin 270°$

Find the values of the other functions, given

7. $\tan \theta = \infty$.
8. $\sin \theta = 0$.
9. $\cos \theta = -1$.

29. Reduction to an Acute Angle. In numerical applications it is sometimes necessary to use angles greater than 90°, but the table of trigonometric functions gives the values only for positive acute angles. Hence it is necessary to be able to express any function of a given angle, no matter what its size, in terms of a function of an acute angle.

The process of expressing a function of θ ($\theta > 90°$) in terms of a function of an acute angle is called *reduction to an acute angle.* There is no loss of generality in assuming θ positive since by definition a function of a negative angle can be expressed readily in terms of a function of a positive angle (§ 25).

We shall give two methods of reduction to an acute angle: first, by the use of the relations between the functions of θ and $\theta - 90°$ (§ 30); and second, by expressing the relation between any function of a given angle A (either positive or negative) and the same function of the acute angle which the terminal side of A makes with the x-axis (§ 32).

30. First Method of Reduction to an Acute Angle—Functions of θ and $\theta - 90°$. The first step in the first method of reduction to an acute angle is to express the functions of θ in terms of the functions of $\theta - 90°$.

Consider first the case in which the terminal side of θ lies in the second quadrant. The appropriate congruent right triangles from which to define the functions of θ and $\theta - 90°$ are shown in Fig. 48.

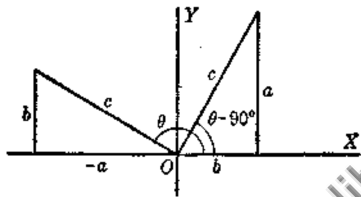


FIG. 48

The letters a, b, c are used to denote positive lengths. From Fig. 48, by definition, we have

(1)
$$\begin{aligned}
\sin\theta &= \frac{b}{c} = \cos(\theta - 90°),\\
\cos\theta &= -\frac{a}{c} = -\sin(\theta - 90°),\\
\tan\theta &= -\frac{b}{a} = -\cot(\theta - 90°),\\
\csc\theta &= \frac{c}{b} = \sec(\theta - 90°),\\
\sec\theta &= -\frac{c}{a} = -\csc(\theta - 90°),\\
\cot\theta &= -\frac{a}{b} = -\tan(\theta - 90°).
\end{aligned}$$

By drawing figures analogous to Fig. 48 and showing θ in the remaining quadrants, we see readily that formulas (1) hold when θ is any angle.

The formulas (1) may be expressed in words as follows: *A trigonometric function of any given angle θ is numerically equal to the co-function of the angle that is smaller by 90°, that is, $\theta - 90°$. The sign, however, will change for all the functions except* **sin θ** *and* **csc θ**

EXERCISES

Express each of the following functions in terms of functions of angles less by 90°.

1. tan 318°

SOLUTION. By (1), we have tan 318° = − cot (318° − 90°) = − cot 228°.

2. cos 170°
3. sin 220°
4. sec 410°
5. cot 500°
6. csc 751°
7. sin 1601°

Express each of the following functions in terms of functions of angles less by 2·90° by applying formulas (1) twice.

8. sin 216°

SOLUTION. By (1) we have

$$\sin 216° = \cos 126° = -\sin 36°.$$

9. sin 255°
10. cos 304°
11. tan 225°
12. cot 334°
13. sec 560°
14. csc 1141°

Express each of the following functions in terms of functions of angles less by 3·90°.

15. sin 329° **16.** cos 350° **17.** tan 1218°

By the repeated use of formulas (1) express each of the following functions in terms of the functions of a positive acute angle, and find their values from table, pp. 16–17.

18. tan 279°

SOLUTION. tan 279° = − cot 189° = tan 99° = − cot 9° = − 6.314

19. cos 153° 5′
20. sin 235° 26′
21. sec 120°
22. cot 300°
23. csc 330°
24. cos 321°
25. sin (− 100°)

SOLUTION. From § 25 it is clear that sin (− 100°) = sin 260°.
Then sin 260° = cos 170° = − sin 80° = − 0.985

26. cos (− 40°) **27.** tan (− 125°) **28.** cos (− 200°)

By applying formulas (1) show that

29. $\sin (180° + A) = -\sin A$.
30. $\cos (180° + A) = -\cos A$.
31. $\cot (270° + A) = -\tan A$.
32. $\cos (90° + A) = -\sin A$.

Give the values of the sine, cosine, and tangent of each of the following angles, without using the tables:

33. 225°
34. 330°
35. 300°
36. 240°
37. 120°
38. 210°

39. Express sin θ in terms of a function of $\theta - 2\cdot 90°$, of $\theta - 3\cdot 90°$, of $\theta - 4\cdot 90°$, of $\theta - 5\cdot 90°$.

40. In Exercise 39, is sin θ expressed in terms of a sine function when the angle is reduced, (1) by an odd multiple of 90°, (2) by an even multiple of 90°?

31. Working Rule for Reduction to Acute Angles. From the repeated use of formulas (1) and from certain exercises in the preceding article the student has, no doubt, noticed that there is a relation between the functions of θ ($\theta > 90°$) and of $\theta - n \cdot 90°$ (n a positive integer). Since the angle $\theta - n \cdot 90°$ can always be made an acute angle by the proper choice of n, it is possible to state the relations between the functions of θ and $\theta - n \cdot 90°$ in the form of a *working rule:*

I. A function of θ is numerically equal to the co-function of the acute angle $\theta - n \cdot 90°$ if n is odd.

II. A function of θ is numerically equal to the same function of the acute angle $\theta - n \cdot 90°$ if n is even.

III. The sign to be prefixed to the result is the sign of the given function of θ in the quadrant to which θ belongs.

EXAMPLE. Express tan 310° as a function of an acute angle.

SOLUTION. In this example $\tan 310° = -\cot (310° - 3 \cdot 90°) = -\cot 40°$. The co-function is used since $n = 3$ is odd, and the negative sign is prefixed since tan 310° is negative, 310° being in the fourth quadrant.

EXERCISES

Reduce each of the following to a function of an acute angle by means of the "working rule."

1. cos 279°
2. tan 153°
3. sin 886°
4. sec 351°
5. cot 688°
6. csc 972°

32. Second Method of Reduction to Acute Angles. A second method of reduction to acute angles, which is readily grasped by the study of examples, will probably be found very attractive and practical by those who like an almost purely geometric method.

EXAMPLE 1. By drawing a figure, express sin 110°, cos 110°, tan 110° as functions of an acute angle.

SOLUTION. Place the angle 110° in standard position (Fig. 49). Examine the values of the functions of the acute angle 70° which the terminal line OP makes with the x-axis.

Then, by definition, the numerical values of sin 110°, cos 110°, tan 110° are respectively equal to the values of sin 70°, cos 70°,

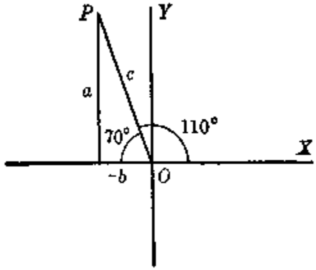


FIG. 49 (a, b, c positive)

tan 70°. Since the sine of an angle in the second quadrant is positive, we have

$$\sin 110^\circ = \sin 70^\circ.$$

Since the cosine and tangent of an angle in the second quadrant are each negative, we have

$$\cos 110^\circ = -\cos 70^\circ, \quad \tan 110^\circ = -\tan 70^\circ.$$

EXAMPLE 2. By drawing a figure express sin (− 153°), cos (− 153°), tan (− 153°) as functions of an acute angle.

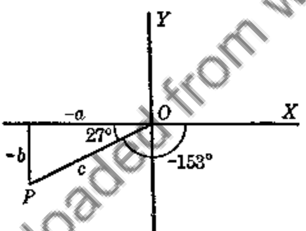


FIG. 50 (a, b, c positive)

SOLUTION. Place the angle − 153° in standard position (Fig. 50). Examine the values of the functions of the acute angle 27° which the terminal line OP makes with the x-axis. Then, by definition, the numerical values of sin (− 153°), cos (− 153°), tan (− 153°) are equal, respectively, to the values of sin 27°, cos 27°, tan 27°. Since the sine and cosine of an angle in the third quadrant are each negative

$$\sin(-153^\circ) = -\sin 27^\circ, \quad \cos(-153^\circ) = -\cos 27^\circ.$$

Since the tangent of an angle in the third quadrant is positive,

$$\tan(-153^\circ) = \tan 27^\circ.$$

EXERCISES

By means of a figure express each of the following functions in terms of functions of an acute angle.

1. sin 155°,	cos 200°,	tan 301°.
2. cot 471°,	sec 562°,	csc 671°.
3. sin 317°,	cos 457°,	tan 567°.
4. cot 1295°,	sec 1365°,	csc 2265°.
5. sin (− 147°),	cos (− 190°),	tan (− 309°).
6. cot (− 770°),	sec (− 860°),	csc (− 950°).

33. Functions of θ and $-\theta$. In many theoretical discussions, it is found desirable to know the relations between the functions of θ and $-\theta$, where θ is any angle. To derive these relations, consider first the case where θ and $-\theta$ terminate in the second and third quadrants respectively. The appropriate congruent right triangles from which to define the functions of $-\theta$ and θ are shown in Fig. 51.

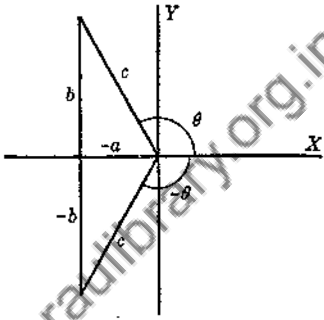


Fig. 51 (a, b, c positive)

By definition, we have

$$(2)\begin{cases} \sin(-\theta) = -\dfrac{b}{c} = -\sin\theta, & \csc(-\theta) = -\dfrac{c}{b} = -\csc\theta, \\ \cos(-\theta) = -\dfrac{a}{c} = \cos\theta, & \sec(-\theta) = -\dfrac{c}{a} = \sec\theta, \\ \tan(-\theta) = \dfrac{-b}{-a} = -\tan\theta, & \cot(-\theta) = \dfrac{-a}{-b} = -\cot\theta. \end{cases}$$

From figures analogous to Fig. 51 it may be shown that relations (2) hold for θ terminating in any quadrant. Hence, we may state the following theorem.

Theorem. *The functions of $-\theta$ are equal to the same-named functions of θ, the signs being opposite for all functions except the cosine and secant.*

EXERCISES

Express each of the following as functions of positive acute angles by first using relations (2) (§ 33).

1. sin (− 120°)

Solution. From (2), we have

$$\sin(-120^\circ) = -\sin 120^\circ = -\cos(120^\circ - 90^\circ) = -\cos 30^\circ.$$

2. sin (− 310°)	**3.** cos (− 285°)	**4.** tan (− 195°)
5. sec (− 211°)	**6.** csc (− 1300°)	

By any method express each of the following as functions of acute angles.

7. $\sin(-314° 30')$ 8. $\cos(-301°)$ 9. $\tan(-100° 30' 20'')$
10 $\cot(-1565°)$ 11. $\sec(-95°)$ 12. $\csc(-275° 22')$

From the table, pp. 16–17, find the value of each of the following:

13. $\sin(-125° 30')$ 14. $\cos(-211° 30')$
15. $\tan(-315° 30')$ 16. $\csc(-241°)$

Prove that:

17. $\sin(180° - A) = \sin A.$
18. $\cos(180° - A) = -\cos A.$
19. $\tan(90° - A) = \cot A.$
20. $\cot(270° - A) = \tan A.$
21. $\sec(270° - A) = -\csc A.$
22. $\cos(90° - A) = \sin A.$
23. $\sin(270° - A) = -\cos A.$

24. Draw a figure and derive relations (2), § 33, for $-\theta$ terminating in the fourth quadrant.

MISCELLANEOUS EXERCISES

1. Find the angle generated by the hour hand of a watch in 5 hours, 40 minutes.

2. Find the angle generated by the minute hand of a watch in 2 hours, 36 minutes.

3. Find the angle generated by the second hand of a watch in 5 hours, 15 minutes, 40 seconds.

4. To what quadrant does each of the following angles belong: 457°, −875°, 380°, 492°, −800°, $15° + 5 \cdot 90°$, $1757° - 6 \cdot 90°$?

5. To what quadrant do the interior (exterior) angles of a regular polygon of 18 sides belong?

6. In a circle of radius 12 feet, how long an arc does a central angle of 210° subtend ($\pi = 22/7$)?

7. Given $\sin\theta = 12/13$, find $\cos\theta$, $\tan\theta$, $\sec\theta$.

8. Given $\sec\theta = 2$, find $\sin\theta$, $\cos\theta$, $\cot\theta$.

9. Given $\tan\theta = 3/4$, $\sin\theta < 0$, find $\sin\theta$, $\cos\theta$, $\csc\theta$.

10. Given $\cos\theta = -7/25$, $\tan\theta < 0$, find $\sin\theta$, $\tan\theta$, $\csc\theta$.

Evaluate the following:

11. $\sin(-120°) + 3\cot 315° - 2\cos 300°$
12. $\tan 120° + 3\cot(-240°) + \sec(-300°) + 6\sin 270°$
13. $\tan 225° \sin 930° \cos(-600°)$
14. $\tan 135° + 4\cot 225° - 5\cos 180°$
15. $5\sin 0° - 6\cos 90° + \tan 180° + \csc 270° + \sec 180°$

16. If $\tan\theta = -a$, and if θ is in the fourth quadrant, find the remaining functions of θ.

17. Given $\sin\theta = l$, and θ in the second quadrant, find the remaining functions.

18. If $\cos\theta = -40/41$, and if θ is in the third quadrant, find the remaining functions.

19. Evaluate

$$\frac{\sin 420° \tan(-210°)}{\csc(-390°) + 3\cot 585°}.$$

20. Given $\tan\theta = -5/12$, $\sin\theta > 0$, find the six trigonometric functions of $-\theta$.

21. Two wheels of radii 10 inches and 12 inches are connected by a belt. If the larger turns through an angle of 1000°, through what angle will the smaller wheel turn?

22. Prove that $\tan(180° + \theta) = \tan\theta$.

23. Prove that $\cos(180° - A) = -\cos A$.

24. Given that $\sin 115° = n$, find the value of

$$\frac{\tan 205° \sin 335°}{\cos 245°}.$$

25. Given $\cos 320° = a$, find the six trigonometric functions of 140°.

26. Simplify

$$\frac{\sin(-\theta)}{\cos(90° + \theta)} - \frac{\tan(-\theta)}{\cot(90° - \theta)} + \frac{\sec(-\theta)}{\csc(180° - \theta)}.$$

27. Prove that $\cos(270° + \theta) = \sin\theta$.

28. Express as functions of positive acute angles:

$\sin 317°$, $\cos(-800°)$, $\tan 241°$, $\cot(-117°)$, $\sec 295°$, $\csc 1200°$.

Find two positive values of θ less than 360° satisfying the following equations:

29. $\sin\theta = -1/2$.

30. $\cos\theta = \sqrt{3}/2$.

31. $\tan\theta = -\sqrt{3}$.

32. $\cot\theta = 1$.

33. $\sec\theta = -2/\sqrt{3}$.

34. $\csc\theta = \sqrt{2}$.

35. Prove that $a\cos(90° - \theta) + b\cos(90° + \theta) = (a - b)\sin\theta$.

36. Simplify $\tan\theta + \tan(-\theta) - \tan(180° - \theta)$.

37. $\sin 420° \cos 390° + \cos(-300°)\sin(-330°) = ?$

38. $5\sec^2 135° - 6\cot^2 300° = ?$

39. Simplify $(a + b)\cot(90° + x) + (a - b)\tan(90° - x) + (b - a)\cot x$.

40. Express the following as functions of acute angles less than 45°: $\sin 263°$, $\cot 333°$, $\cos 642°\,10'$, $\tan 462°\,15'$, $\cos 284°$.

CHAPTER IV

TRIGONOMETRIC IDENTITIES

34. Meaning of an Identity. It will be recalled from algebra that an ***identity*** is an equality that holds for *all* values of the *unknown* or *unknowns* for which each member of the equality is defined. Thus, $\sin^2\theta + \cos^2\theta = 1$ is an example of an identity that is true for all values of θ; whereas $\tan\theta = \sin\theta/\cos\theta$ is an example of an identity whose members are not defined for $\theta = 90°$ nor for θ equal to an odd multiple of 90°.

EXAMPLE. Explain why $\sin\theta/\cos\theta$ is undefined at $\theta = 90°$.

35. Fundamental Identities. In § 10 we found, by considering certain right triangles, that the following equalities hold for any acute angle.

(1) $\sin^2\theta + \cos^2\theta = 1,$ (5) $\sec\theta = \dfrac{1}{\cos\theta},$

(2) $\sec^2\theta = 1 + \tan^2\theta,$ (6) $\cot\theta = \dfrac{1}{\tan\theta},$

(3) $\csc^2\theta = 1 + \cot^2\theta,$ (7) $\tan\theta = \dfrac{\sin\theta}{\cos\theta},$

(4) $\csc\theta = \dfrac{1}{\sin\theta},$ (8) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}.$

By drawing in each quadrant the appropriate right triangles analogous to those in § 10, it readily follows that the above equalities hold for the angle θ terminating in any quadrant, thereby removing the restriction that the angle θ be an acute angle. Since these equalities hold for all values of the angle θ for which their members are defined (§ 27), they are identities. The eight identities given above are sometimes called the *fundamental trigonometric identities.*

36. Trigonometric Transformations. In higher mathematics it is frequently desirable to transform a trigonometric expression into another form. This may be for the purpose of simplifying the expression, or for reducing it to a form containing one or two functions only. The fundamental identities are useful in making such transformations.

EXAMPLE 1. By using the fundamental identities simplify the expression

$$\frac{\cos\theta}{1-\sin^2\theta}.$$

SOLUTION. From (1), $1 - \sin^2\theta = \cos^2\theta$. Hence

$$\frac{\cos\theta}{1-\sin^2\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{1}{\cos\theta} = \sec\theta.$$

EXAMPLE 2. Express $\dfrac{\sec\theta\sin\theta}{\tan\theta+\cot\theta}$ in terms of $\sin\theta$ only.

SOLUTION. By (5), (7), and (8), we have

$$\frac{\sec\theta\sin\theta}{\tan\theta+\cot\theta} = \frac{\dfrac{1}{\cos\theta}\cdot\sin\theta}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}} = \frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\sin\theta\cos\theta}} = \sin^2\theta.$$

EXERCISES

1. Express $\tan\theta\cos\theta + 1 - \sin\theta\sec\theta\cot\theta$ in terms of $\sin\theta$ only.
2. Express $\dfrac{\tan x - 1}{\tan x + 1}$ in terms of $\sin x$ and $\cos x$ only.
3. Simplify $\dfrac{\sin\theta}{1-\cos^2\theta}$.
4. Simplify $\sec z - \tan z\sin z$.
5. Simplify $\dfrac{\sin\theta-\cos\theta}{\tan\theta\csc\theta-\sec\theta\cot\theta}$.
6. Show that $\cot\theta\cos\theta - \csc\theta(1-2\sin^2\theta)$ can be reduced to $\sin\theta$.
7. Show that $\sec^4\theta - \tan^4\theta$ can be transformed into $1 + 2\tan^2\theta$.
8. Express $\sin^4 y - \cos^4 y$ in terms of $\sin y$ only, $\cos y$ only.
9. Show that $(\sin\theta+\cos\theta)^2$ can be transformed into $1 + 2\sin\theta\cos\theta$.
10. Simplify $(1-\cos^2\theta)(1+\cot^2\theta)$.
11. Simplify $(\sin\theta+\cos\theta)^2 + (\sin\theta-\cos\theta)^2$.
12. Simplify $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$.

13. Express $\frac{\sec x - \csc x}{\sec x + \csc x}$ in terms of $\tan x$ only.

14. Show that $(\tan x + \sec x)^2$ can be transformed into $\frac{1 + \sin x}{1 - \sin x}$.

15. Show that $(\csc x - \cot x)^2$ can be transformed into $\frac{1 - \cos x}{1 + \cos x}$.

16. Show that $(1 + \tan y)^2 + (1 + \cot y)^2$ can be transformed into $(\sec y + \csc y)^2$.

37. Further Examples of Identities. The reduction formulas of §§ 30 and 33 are examples of important trigonometric identities. An indefinitely large number of trigonometric identities of greater or less importance can be established by means of the fundamental identities (§ 35). We shall give examples and exercises to illustrate methods of establishing a number of such identities.

EXAMPLE 1. Prove that

$$\csc \theta + \cot \theta = \frac{\cos^2 \theta + \cos \theta + \sin^2 \theta}{\sin \theta}$$

by transforming the second member into the first.

SOLUTION.

$$\frac{\cos^2 \theta + \cos \theta + \sin^2 \theta}{\sin \theta} = \frac{1 + \cos \theta}{\sin \theta} = \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \csc \theta + \cot \theta.$$

EXAMPLE 2. Prove that $\tan \theta + \cot \theta = \frac{\sec \theta}{\sin \theta}$ by transforming each member to the same expression.

SOLUTION.

$$\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta}.$$

$$\frac{\sec \theta}{\sin \theta} = \frac{\frac{1}{\cos \theta}}{\sin \theta} = \frac{1}{\sin \theta \cos \theta}.$$

EXAMPLE 3. Prove that

$$(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x.$$

SOLUTION. Expressing all functions in terms of $\sin x$ and $\cos x$, we have

First Member

$$(\sin x + \cos x)\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= (\sin x + \cos x)\left(\frac{1}{\sin x \cos x}\right)$$

$$= \frac{\sin x + \cos x}{\sin x \cos x}.$$

Second Member

$$\frac{1}{\cos x} + \frac{1}{\sin x}$$

$$= \frac{\sin x + \cos x}{\sin x \cos x}.$$

EXERCISES

Prove the following identities by reducing the first member to the second.

1. $\cos^2\theta \tan^2\theta + \sin^2\theta \cot^2\theta = 1.$
2. $\sin\theta(\cot\theta + 2)(2\cot\theta + 1) = 2\csc\theta + 5\cos\theta.$
3. $(\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha) = \sec\alpha + \csc\alpha.$
4. $\dfrac{\cos\theta}{1 - \tan\theta} + \dfrac{\sin\theta}{1 - \cot\theta} = \sin\theta + \cos\theta.$
5. $\dfrac{1 + \tan^2 x}{\csc^2 x} = \tan^2 x.$
6. $\dfrac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2.$
7. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2.$
8. $\dfrac{\sec\theta + 1}{\tan\theta} - \dfrac{\tan\theta}{\sec\theta - 1} = 0.$
9. $\dfrac{\sec^2\theta}{\sec^2\theta - 1} = \csc^2\theta.$
10. $\dfrac{\cos A}{1 + \sin A} = \dfrac{1 - \sin A}{\cos A}.$
11. $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = 2\csc\theta.$
12. $\tan\theta + (\cos^2\theta - \sin^2\theta)\sec\theta\csc\theta = \cot\theta.$

Prove the following identities by reducing the second member to the first.

13. $\tan A - \cot A = \dfrac{1 - 2\cos^2 A}{\sin A \cos A}.$
14. $2\csc^2\theta = \dfrac{1}{1 - \cos\theta} + \dfrac{1}{1 + \cos\theta}.$
15. $\tan^2 x \sin^2 x = \tan^2 x - \sin^2 x.$
16. $\tan^2\theta + 1 = \csc^2\theta \tan^2\theta.$
17. $\sec^2\theta \csc^2\theta = \sec^2\theta + \csc^2\theta.$
18. $\csc x + \cot x = \dfrac{\sin x}{1 - \cos x}.$

19. $\tan^2\theta + \cot^2\theta + 2 = \sec^2\theta \csc^2\theta.$

20. $(\tan\theta - \cot\theta)^2 = \sec^2\theta + \csc^2\theta - 4.$

21. $\dfrac{1}{1 - \tan B} = \dfrac{\cos B}{\cos B - \sin B}.$

22. $\dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}.$

Prove the following identities by reducing each member to the same expression.

23. $\cos x \csc x \tan x = 1 - \cos x + \cot x \sin x.$

24. $\sqrt{\dfrac{\sec^2\theta - 1}{1 - \sin^2\theta}} = \dfrac{\sec\theta}{\cot\theta}.$

25. $2 + \dfrac{\sin^4\theta + \cos^4\theta}{\sin^2\theta\cos^2\theta} = \sec^2\theta\csc^2\theta.$

26. $\cos^2 A \tan A = \dfrac{2\sin A}{\sec A + \cos A + \sin^2 A \sec A}.$

27. $\dfrac{\cot\theta\cos\theta}{\cot\theta + \cos\theta} = \dfrac{\cot\theta - \cos\theta}{\cot\theta\cos\theta}.$

28. $\tan B + \cot B = \dfrac{\csc^2 B + \sec^2 B}{\csc B \sec B}.$

29. $\sec^4\theta(1 - \sin^4\theta) = \sec^2\theta + \tan^2\theta.$

30. $\cot^4 x + \cot^2 x = \csc^4 x - \csc^2 x.$

31. $(1 + \sec\theta)(1 - \cos\theta) = \tan\theta\sin\theta.$

32. $\dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\cot\beta - \cot\alpha}{\cot\alpha\cot\beta + 1}.$

Prove each of the following identities by any method.

33. $(\tan A - \cot A)\sin A\cos A = \sin^2 A - \cos^2 A.$

34. $\tan^4 x + \tan^2 x = \sec^4 x - \sec^2 x.$

35. $\cot\theta - \sec\theta\csc\theta(1 - 2\sin^2\theta) = \tan\theta.$

36. $1 + \dfrac{4\tan^2\theta}{(1 - \tan^2\theta)^2} = \dfrac{1}{(\cos^2\theta - \sin^2\theta)^2}.$

37. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2(1 + \tan^2\theta).$

38. $(1 + \tan x + \cot x)(\sin x - \cos x) = \dfrac{\sec x}{\csc^2 x} - \dfrac{\csc x}{\sec^2 x}.$

39. $2\tan^2\theta + \tan^4\theta = \sec^4\theta - 1.$

40. $\sin^2 A \sec^2 A = \sec^2 A - 1.$

41. $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta.$

42. $\dfrac{1}{1 + \cos^2\theta} = \dfrac{\sec^2\theta}{\tan^2\theta + 2}.$

43. $\dfrac{2\tan x}{1 - \tan^2 x} = \dfrac{2}{\cot x - \tan x}.$

Find the positive values of θ less than 360° for which one or both members of each of the identities in Exercises 44–48 are undefined.

44. $2\csc^2\theta = \dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta}$.

SOLUTION. When $\theta = 0$, $\csc\theta$ is undefined, and the denominator $1 - \cos\theta$ is zero. When $\theta = 180°$, $\csc\theta$ is undefined, and the denominator $1 + \cos\theta$ is zero. Hence, both members of the identity are undefined when $\theta = 0°$, or 180°.

45. $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$.

46. $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$.

47. $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$.

48. $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2}{\cot\theta - \tan\theta}$.

CHAPTER V

VARIATION OF THE TRIGONOMETRIC FUNCTIONS. LINE VALUES. GRAPHS

38. Variation in Values of the Functions. The variations in the values of a trigonometric function, say of $\sin \theta$, as θ increases from 0° to 90° can be followed fairly well in a table such as that on pages 16–17.

The line values and the graphs of the functions to be discussed in the present chapter are of much assistance not only in following changes in values of the functions from 0° to 90°, but also in tracing the variations throughout the whole range of values of the angle from 0° to 360°.

39. Line Values. Let θ be any angle (Figs. 52–55). With the origin of coordinates, O, as a center and with a radius $r = 1$ unit, draw a circle (often called a *unit* circle), cutting the initial side of θ at A and the terminal side at P.

Draw MP perpendicular to the x-axis, and draw the tangents AT and BT' to the circle at the points A and B as shown. Extend these tangents to meet the terminal side of θ at T and T'. Then in each of the four quadrants (Figs. 52–55), we may write, since $OP = 1$,

$$\sin \theta = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

$$\cos \theta = \frac{OM}{OP} = \frac{OM}{1} = OM,$$

$$\tan \theta = \frac{MP}{OM} = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

$$\csc \theta = \frac{OP}{MP} = \frac{OT'}{OB} = \frac{OT'}{1} = OT',$$

$$\sec \theta = \frac{OP}{OM} = \frac{OT}{OA} = \frac{OT}{1} = OT,$$

$$\cot \theta = \frac{OM}{MP} = \frac{BT'}{OB} = \frac{BT'}{1} = BT'.$$

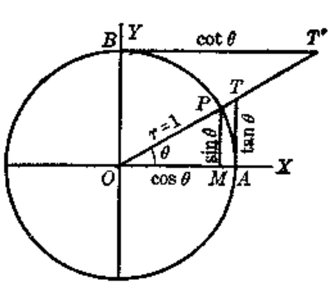


FIG. 52

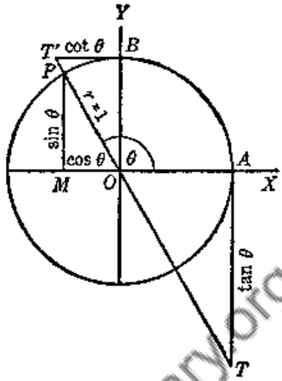


FIG. 53

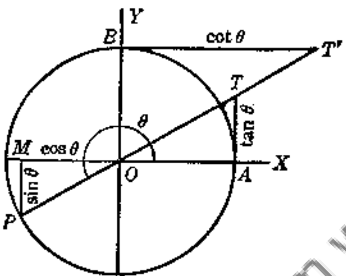


FIG. 54

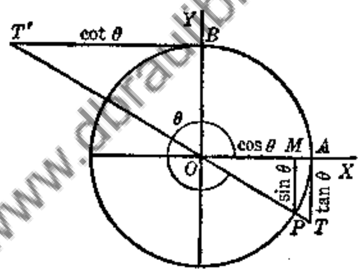


FIG. 55

We thus define the line values of the six functions to be

$$\sin\theta = MP, \qquad \csc\theta = OT',$$
$$\cos\theta = OM, \qquad \sec\theta = OT,$$
$$\tan\theta = AT, \qquad \cot\theta = BT'.$$

That is, each trigonometric function is given both in magnitude and sign by a line when the unit of length is the radius of the unit circle, it being understood that OT (also OT') is negative when its direction from O is opposite to that of the terminal line OP.

By following the variations of the appropriate line values in Figs. 52, 53, 54, 55, as θ changes from 0° to 360°, we may form a useful general idea of the variation of a function as θ increases.

Tracing Variations in sin θ. By observing the changes in length of MP as θ increases from 0° to 360°, we draw the following inferences:

As θ increases from 0° to 90° (Fig. 52), $MP = \sin\theta$ starts at 0 and increases to 1.

As θ increases from 90° to 180° (Fig. 53), $MP = \sin\theta$ starts from 1 and decreases to 0.

As θ increases from 180° to 270° (Fig. 54), $MP = \sin\theta$ starts from 0 and decreases to -1.

As θ increases from 270° to 360° (Fig. 54), $MP = \sin\theta$ starts from -1 and increases to 0.

Tracing Variations in cos θ. By observing the changes in the length of OM as θ increases from 0° to 360° we infer that:

As θ increases from 0° to 90° (Fig. 52), $OM = \cos\theta$ starts at 1 and decreases to 0.

As θ increases from 90° to 180° (Fig. 53), $OM = \cos\theta$ starts from 0 and decreases to -1.

As θ increases from 180° to 270° (Fig. 54), $OM = \cos\theta$ starts from -1 and increases to 0.

As θ increases from 270° to 360° (Fig. 55), $OM = \cos\theta$ starts from 0 and increases to 1.

Tracing Variations in tan θ. By observing the changes in AT as the angle θ increases from 0° to 360°, we may trace the variation of $\tan\theta$ and find that:

As θ increases from 0° to 90° (Fig. 52), $AT = \tan\theta$ increases from 0 to ∞.

As θ increases from 90° to 180° (Fig. 53), $AT = \tan\theta$ increases from $-\infty$ to 0.

As θ increases from 180° to 270° (Fig. 54), $AT = \tan\theta$ increases from 0 to ∞.

As θ increases from 270° to 360° (Fig. 55), $AT = \tan\theta$ increases from $-\infty$ to 0.

The variation in the other functions may be similarly traced from Figs. 52, 53, 54, 55, and such tracing will now be given as exercises.

EXERCISES

1. Trace and describe the changes in the value of $\cot\theta$ as θ increases from 0° to 90°, 90° to 180°, 180° to 270°, 270° to 360°, by following the changes in BT' (Figs. 52–55).

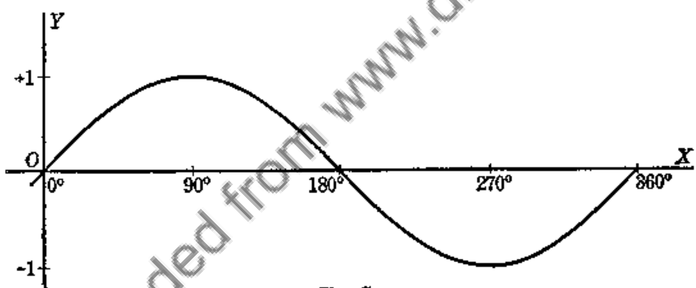
2. Same as Exercise 1 but for $\sec\theta$ instead of $\cot\theta$.

3. Same as Exercise 1 but for $\csc\theta$ instead of $\cot\theta$.

40. Graphs of sin x, cos x, and tan x. Let x be a variable angle. To obtain a suitable set of values for plotting the graph of a function, say of $y = \sin x$, it is usually convenient to take for x the angles 0°, 30°, 45°, 60°, 90°, 120°, $\cdots$, and prepare a table of values of y that correspond to assigned values of x. Thus:

x	$y = \sin x$	x	$y = \sin x$	x	$y = \sin x$	x	$y = \sin x$
0°	0.00	90°	1.00	180°	0.00	270°	-1.00
30°	$\frac{1}{2} = 0.50$	120°	$\frac{1}{2}\sqrt{3} = 0.87$	210°	$-\frac{1}{2} = -0.50$	300°	$-\frac{1}{2}\sqrt{3} = -0.87$
45°	$\frac{1}{2}\sqrt{2} = 0.71$	135°	$\frac{1}{2}\sqrt{2} = 0.71$	225°	$-\frac{1}{2}\sqrt{2} = -0.71$	315°	$-\frac{1}{2}\sqrt{2} = -0.71$
60°	$\frac{1}{2}\sqrt{3} = 0.87$	150°	$\frac{1}{2} = 0.50$	240°	$-\frac{1}{2}\sqrt{3} = -0.87$	330°	$-\frac{1}{2} = 0.50$

Sine Curve. On OX, Fig. 56, mark off an angle scale for x of some convenient length so that 360° can be easily shown on the paper that you are using. On OY mark off a scale for y with a convenient length to represent one unit value of y.



Sine Curve
$y = \sin x$

FIG. 56

While it is not essential to have a relation between the scales for x and y, it turns out later, for certain purposes, that the length taken on OX for 90° should be slightly more than the unit length taken on OY multiplied by 3/2. Moreover, the trigonometric curves seem to look well if plotted with about these relative scales for x and y. Locate points on Fig. 56 by using the pairs of values (x, y) in the above table as the coördinates of points. By drawing a smooth curve through the points, we obtain the wave-shaped curve of Fig. 56, called the ***sine curve,*** or the ***graph of sin x.***

If greater accuracy is desired than is assured by the number of plotted points, we may well increase the number of points by plotting values at smaller intervals, say at intervals of only 15° on x, and obtain the values of the function from the table on pp. 16–17.

The sine curve from $x = 0°$ to $360°$ would clearly be repeated if we should continue the curve for values of x from 360° to 720°. Indeed, the curve may be continued without bound both to the right and to the left in Fig. 56. On account of this repetition of values, the sine function is said to be a ***periodic*** function. The period of $\sin x$ is 360° as may be seen from the curve or from $\sin (x + 360°) = \sin x$.

On account of its simple wave form, the sine curve has many applications in wave motion.

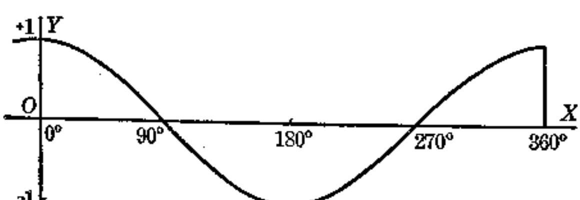
The greatest distance of the sine curve, $y = \sin x$, from the x-axis is called the ***amplitude*** of the curve or wave.

The ***wave length*** is equal to the period of the trigonometric function that represents the wave.

EXERCISES

1. What is the value of the amplitude of the sine curve in Fig. 56?
2. How far is the lowest point of the sine curve (Fig. 56) below the x-axis?
3. For what values of x is $\sin x = 0$?
4. Give a verbal description of the variations of the sine function from your observation of the curve for $y = \sin x$ in Fig. 56 as x increases from: (a) 0° to 90°; (b) 90° to 180°; (c) 180° to 270°; (d) 270° to 360°.
5. What is the amplitude of $y = 2 \sin x$?

Cosine Curve. To plot the graph of $y = \cos x$, we first prepare a table of corresponding values of x and y, plot these values, and draw a smooth curve through them as shown in Fig. 57.



Cosine Curve
$y = \cos x$

Fig. 57

EXERCISES

1. Since $\sin(x + 90°) = \cos x$, how does the curve $y = \cos x$ differ from that of $y = \sin x$?

2. For what values of x does $\cos x$ have its smallest algebraic value? What is this value?

3. For what values of x is $\cos x = 0$?

4. What is the amplitude of the cosine curve?

5. Give a verbal comparison of the cosine curve with the sine curve.

6. What is the period of $\cos x$?

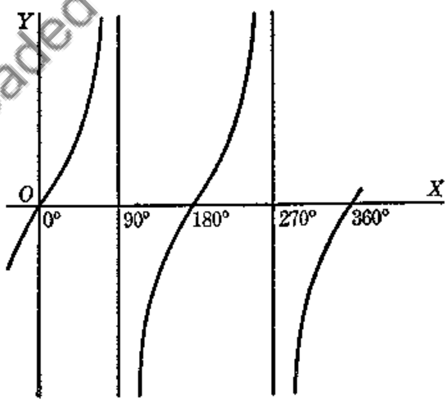
Tangent Curve. On account of the fact that $\tan x$ changes rapidly when x is near 90° and 270°, we suggest for purposes of plotting

$$y = \tan x$$

the following values at intervals of 15° from the table on pages 16–17 rather than at wider intervals used to plot $\sin x$ and $\cos x$.

x	$y = \tan x$	x	$y = \tan x$	x	$y = \tan x$	x	$y = \tan x$
0°	0.00	90°	∞	180°	0.00	270°	∞
15°	0.27	105°	− 3.73	195°	0.27	285°	− 3.73
30°	0.58	120°	− 1.73	210°	0.58	300°	− 1.73
45°	1.00	135°	− 1.00	225°	1.00	315°	− 1.00
60°	1.73	150°	− 0.58	240°	1.73	330°	− 0.58
75°	3.73	165°	− 0.27	255°	3.73	345°	− 0.27

With these values the graph of the curve may be drawn as in Fig. 58.



Tangent Curve
$y = \tan x$

FIG. 58

EXERCISES

1. Does the tangent have a finite amplitude?

2. What is the period of tan x?

3. Is the tangent curve continuous? If not, for what angles is it discontinuous?

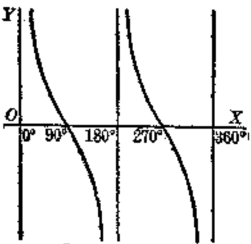
4. Does the tangent function always increase as the angle increases? Does tan x increase continuously as x changes continuously from 89° to 91°? Explain from the curve.

5. Plot $y = \sin(x + 90°)$.

6. Plot $y = \cos(x + 90°)$.

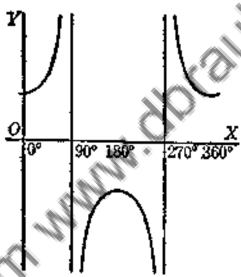
7. Plot $y = \tan(x + 90°)$.

41. Graphs of cot *x*, sec *x*, csc *x*. In a manner similar to that used for the three principal functions, we may draw the graphs of cot x, sec x, and csc x, as shown in Figs. 59–61.



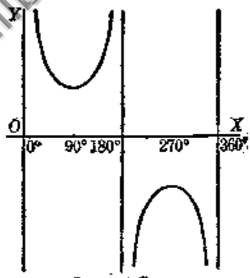
Cotangent Curve
$y = \cot x$

FIG. 59



Secant Curve
$y = \sec x$

FIG. 60



Cosecant Curve
$y = \csc x$

FIG. 61

EXERCISES

1. Does the cosecant curve have a finite amplitude? The secant? The cotangent?

2. How near does the curve $y = \sec x$ come to the x-axis?

3. How near does the curve $y = \csc x$ come to the x-axis?

4. Does cot x decrease continuously as x changes continuously from 179° to 181°?

5. At what values of x has $y = \cot x$ discontinuities?

CHAPTER VI

LAW OF SINES—LAW OF COSINES—APPLICATIONS

42. Introduction. In Chapter II, certain problems of a practical nature were solved. The solutions of those problems were made to depend upon the solutions of right triangles. In the present chapter, we derive formulas by means of which practical problems involving oblique triangles may be solved. For this purpose, we shall make much use of the law of sines and law of cosines.

43. Law of Sines. *In any triangle ABC,*

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

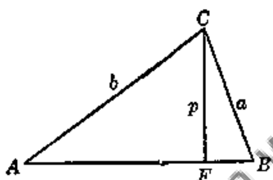


FIG. 62

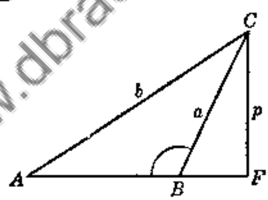


FIG. 63

PROOF. In the triangle ABC, Fig. 62, or Fig. 63, draw the altitude, $CF = p$, forming the right triangles AFC and BFC. From these right triangles, in either figure, we have

$$\frac{p}{b} = \sin A, \qquad \frac{p}{a} = \sin B.$$

Dividing p/b by p/a, we get

(1) $$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Since a and b are any two sides of the triangle ABC, we may write

(2) $$\frac{a}{c} = \frac{\sin A}{\sin C},$$

(3) $$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

These results are usually written in the more symmetrical form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In this form, the law of sines can be stated as follows: *In any triangle the sides are proportional to the sines of the opposite angles.*

We shall now give some numerical examples to be solved by the use of the law of sines. In the computations of this chapter, we use the four-place tables of trigonometric functions, Table II, pp. 21–26. Since we assume all our data exact for purposes of computation, distances computed by means of values from Table II will be reported to four significant figures. Angles will be reported to the nearest minute.

EXAMPLE 1. Given $A = 72° 10'$, $C = 41° 50'$, $c = 60$, find a.

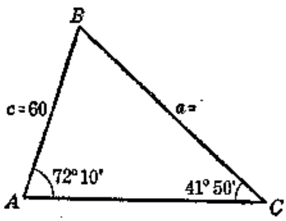


FIG. 64

SOLUTION. The known parts are indicated in Fig. 64. From (2), we have $a = c \sin A/\sin C$. From Table II, pp. 21–26, we find $\sin 72° 10' = 0.9520$, and $\sin 41° 50' = 0.6670$. Substituting, we have

$$a = \frac{(60)(0.9520)}{(0.6670)} = 85.64$$

EXAMPLE 2. Given $B = 58°$, $C = 64° 14'$, $c = 20$, find b.

SOLUTION. The known parts are indicated in Fig. 65. From (3), we have

(a) $$b = \frac{c \sin B}{\sin C}.$$

From Table II, pp. 21–26, we find

(b) $$\sin 58° = 0.8480.$$

Since the angle $64° 14'$ is not found in Table II, we use a process called ***interpolation*** to find the approximate value of $\sin 64° 14'$. From Table II, we find

$$\sin 64° 20' = 0.9013,$$
$$\sin 64° 10' = 0.9001,$$

and note that for the difference of $10'$ in the angle there is a ***tabular*** difference of 0.0012 in the values of the sine function.

Since 64° 14′ is 4/10 of the way from 64° 10′ to 64° 20′, we assume that its sine is 4/10 of the way from sin 64° 10′ to sin 64° 20′, that is, 4/10 of the way from 0.9001 to 0.9013. Hence the correction to be added to 0.9001 is 4/10 of 0.0012 = 0.0005 to the nearest unit in the fourth decimal place; that is,

(c) $\sin 64° 14' = 0.9001 + 0.0005$
$= 0.9006$

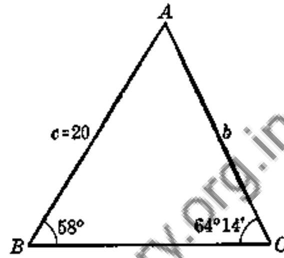


FIG. 65

Substituting from (b) and (c) in (a), we have

$$b = \frac{(20)(0.8480)}{(0.9006)} = 18.83$$

EXAMPLE 3. Given $B = 67° 32'$, $b = 294$, $c = 189$, find C.

SOLUTION. The known parts are indicated in Fig. 66. From (3), we have

$$\frac{\sin C}{\sin B} = \frac{c}{b},$$

that is,

$$\sin C = \frac{c \sin B}{b}.$$

By interpolation, we find

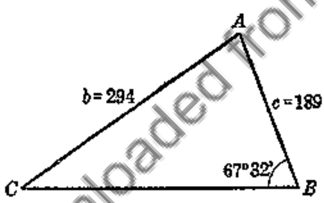


FIG. 66

$$\sin 67° 32' = 0.9241$$

Substituting, we get

$$\sin C = \frac{(189)(0.9241)}{(294)} = 0.5941$$

Then

$$C = 36° 27'.$$

The value of C is found from Table II, page 26, by interpolation. The process of interpolation is ordinarily carried out in an informal manner and may be explained as follows:

$$\sin 36° 20' = 0.5925,$$
$$\sin C = 0.5941,$$
$$\sin 36° 30' = 0.5948$$

The difference between 0.5925 and 0.5941 is 0.0016 and the difference between 0.5925 and 0.5948 is 0.0023. Hence the number 0.5941 lies 16/23 of the way from 0.5925 to 0.5948.

We then assume that the angle C lies 16/23 of the way from 36° 20′ to 36° 30′. The correction to be added to the angle 36° 20′ is 16/23 of 10′ = 7′ to the nearest minute, that is,

$$C = 36° \, 20' + 7' = 36° \, 27'.$$

EXERCISES

1. Given $b = 40$, $A = 67° \, 25'$, $C = 55° \, 49'$, find c.
2. Given $c = 76$, $B = 37° \, 18'$, $C = 35° \, 51'$, find a.
3. Given $a = 120$, $A = 75° \, 11'$, $B = 65° \, 31'$, find b.
4. Given $b = 22$, $c = 75$, $C = 32° \, 20'$, find B.
5. Given $b = 353$, $c = 295$, $B = 46° \, 15'$, find C.
6. Given $a = 17$, $c = 25$, $C = 61° \, 56'$, find A.
7. Show that the law of sines may be written in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

8. State the law of sines in words when written as in Exercise 7.

44. Applications of the Law of Sines. The law of sines may be used to compute the remaining parts of any oblique triangle in which we have given one side and two angles (Case I), or two sides and an angle opposite one of them (Case II).

The following examples will illustrate these two cases.

EXAMPLE 1. In order to calculate the distance from an object A to an inaccessible but visible object C, a base line AB was taken 1200 feet long. The angles A and B were found to be 37° 10′ and 75° 41′, respectively. Find the distance AC.

SOLUTION. The known items are indicated in Fig. 67. Since $A + B + C = 180°$, we have

$$C = 180° - A - B = 180° - 37° \, 10' - 75° \, 41' = 67° \, 9'.$$

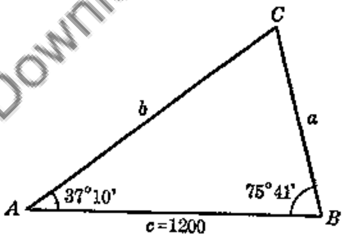


FIG. 67

From (3), § 43, we find

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \text{that is,} \quad b = \frac{c \sin B}{\sin C}.$$

Substituting 1200 for c and the values of $\sin B$ and $\sin C$ from Table II, we have

$$b = \frac{(1200)(0.9690)}{(0.9215)} = 1262 \text{ feet.}$$

EXAMPLE 2. A telephone pole 40 feet high, standing on the side of a hill is braced by a wire running from its top down the hill to a point 35 feet from the foot of the pole. If the pole subtends an angle of 31° 21′ at the point where the wire hits the ground, how long is the wire?

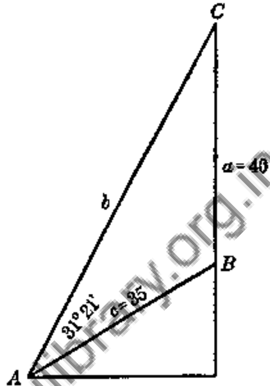


FIG. 68

SOLUTION. The known items are shown in Fig. 68. From (2), § 43, we have

$$\sin C = \frac{c \sin A}{a}.$$

Substituting, we have

$$\sin C = \frac{(35)(0.5202)}{40} = 0.4552$$

Then

$$C = 27° 5', \quad \text{and} \quad A + C = 58° 26'.$$

From (1), § 43, we have

$$b = \frac{a \sin B}{\sin A}.$$

Since $B = 180° - (A + C)$, $\sin B = \sin (A + C)$. Why? Upon substitution, we find

$$b = \frac{(40)(0.8520)}{(0.5202)} = 65.51$$

EXERCISES

1. Find the distance BC in Example 1 above. See Fig. 67.

2. In Example 2 above, find the inclination of the hill (Fig. 68) and the vertical distance of C above A.

3. The distance, $AB = 1000$ feet, between two points along the shore of a lake subtends an angle of 70° at a point C on the opposite shore. If $AC = 800$ feet, how far is B from C? Find the nearest distance from C to AB.

4. The angle of elevation of the top of a mountain from a point at its base is 40° 21′. At a point 1000 feet up the slope of the mountain inclined at a constant angle of 17° 31′, the angle of elevation of the top is 55° 38′. Find the height of the mountain.

5. A boat is steaming N 40° E at a rate of 18 miles per hour. At ten o'clock a radio tower bears N 10° W, at 12 o'clock it bears S 70° W. How far was the ship from the tower at ten o'clock?

6. In Exercise 5, at what time was the ship nearest the tower?

7. Show that the law of sines may be written in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the radius of the circle circumscribing the triangle ABC (Fig. 69).

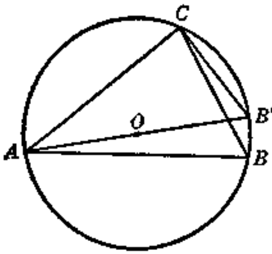


FIG. 69

HINT.

$$\angle B = \angle AB'C. \quad \text{Why?}$$

$$\angle ACB' = 90°. \quad \text{Why?}$$

$$2R \sin AB'C = b. \quad \text{Why?}$$

8. The perimeter of a certain triangle is 100. If $\sin A : \sin B : \sin C = 3 : 2 : 3$, find the sides of the triangle.

9. If the angles of a triangle are in the ratio $1 : 3 : 5$, and the greatest side is 50, find the remaining sides.

10. Show that the length of the internal bisector of the angle C of the triangle ABC may be written in the form

$$\frac{b \sin A}{\sin\left(A + \frac{C}{2}\right)}.$$

11. In the triangle ABC, $c = 100$, $A = 75° \, 18'$, and $B = 50° \, 13'$, calculate the length of the internal bisector of the angle A.

12. In Fig. 70, the line CE bisects the angle C and $AC = DC$. Prove that $\angle BAD = (A - B)/2$, and that $\angle ADB = 90° + C/2$. Apply the law of sines to the triangle ABD and obtain

$$\frac{c}{\cos \frac{C}{2}} = \frac{a - b}{\sin \frac{A - B}{2}}.$$

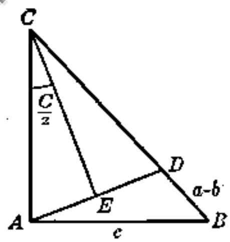


FIG. 70

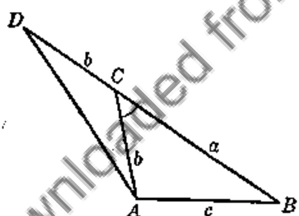


FIG. 71

13. In Fig. 71, it is given that $CD = AC$. Show that $\angle ADC = C/2$, and show also that $\angle BAD = 90° + A/2 - B/2$. Apply the law of sines to the triangle BAD to obtain

$$\frac{c}{\sin \frac{C}{2}} = \frac{a + b}{\cos \frac{A - B}{2}}.$$

This is sometimes called ***Newton's formula.***

The formulas in Exercises 12 and 13 are useful for checking solutions since they contain all six parts of the triangle ABC.

14. From the results in Exercises 12 and 13, obtain the relation

$$\frac{\tan \frac{A - B}{2}}{\tan \frac{A + B}{2}} = \frac{a - b}{a + b}$$

by noting that

$$\cot \frac{C}{2} = \tan \frac{A + B}{2}.$$

The above result is known as the ***law of tangents.***

45. Law of Cosines. *In any triangle ABC,*

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

PROOF. In Fig. 72 and Fig. 73 construct the altitude $BF = p$.

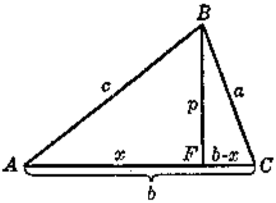


FIG. 72

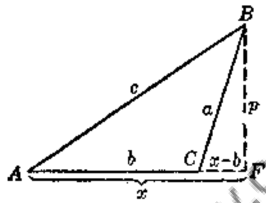


FIG. 73

In the right triangle BFC, we have, from Fig. 72,

$$a^2 = (b - x)^2 + p^2,$$

and, from Fig. 73,

$$a^2 = (x - b)^2 + p^2.$$

Expanding we find

(a) $$a^2 = b^2 + x^2 + p^2 - 2bx.$$

But $p^2 + x^2 = c^2$ and $x = c \cos A$. Why? Hence (a) becomes

(4) $$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A.}$$

Since a is any side of the triangle, we may write also

(5) $$b^2 = a^2 + c^2 - 2ac \cos B,$$
(6) $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Relations (4), (5), and (6) may be put in words as follows. *The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice their product multiplied by the cosine of their included angle.*

EXAMPLE 1. Given $b = 40$, $c = 50$, $A = 65° 18'$, find a.

SOLUTION. To get a, we use formula (4),

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

From Table II, we find, by interpolation, that

$$\cos 65° 18' = 0.4178$$

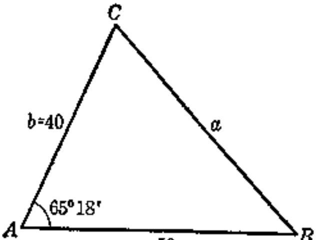


FIG. 74

Substituting, we have (Fig. 74)

$$a^2 = 40^2 + 50^2 - (2)(40)(50)(0.4178)$$
$$= 2429.$$

Hence $a = 49.28$

EXAMPLE 2. Given (see Fig. 75) $a = 250$, $b = 300$, $c = 400$, find C.

SOLUTION. Solving for $\cos C$ in (6), we find

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Then, (Fig. 75),

$$\cos C = \frac{(250)^2 + (300)^2 - (400)^2}{(2)(250)(300)}$$
$$= -0.0500,$$

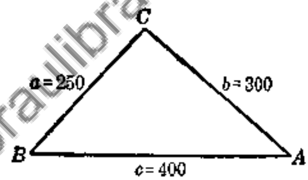


FIG. 75

whence, by interpolation, Table II, p. 22, we find that $87° 8'$ is the acute angle whose cosine is 0.0500. Hence

$$C = 92° 52'. \quad \text{Why?}$$

EXERCISES

1. Given $a = 17$, $b = 20$, $C = 56° 17'$, find c.
2. Given $b = 20$, $c = 25$, $A = 120°$, find a.
3. Given $a = 117$, $c = 120$, $B = 60°$, find b.
4. Given $a = 17$, $b = 20$, $c = 30$, find B.
5. Given $a = 25$, $b = 31$, $c = 29$, find C.
6. Given $a = 200$, $b = 300$, $c = 400$, find A and B.
7. Show that the relations (4), (5), and (6) can be written

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

8. When written as in Exercise 7, state the law of cosines in words.
9. Using Fig. 72 and Fig. 73, show that

$$b = c \cos A + a \cos C,$$

and write similar expressions for a and c.

10. Using the law of cosines show that the square of the length of the median drawn from the vertex B of the triangle ABC is $a^2/2 + c^2/2 - b^2/4$.

11. Find the length of the median to the longest side in Exercise 6 above.

12. In any triangle ABC, if $C = 90°$, $\cos C = 0$, $\cos A = b/c$, $\cos B = a/c$, show that all three relations in Exercise 7 reduce to $c^2 = a^2 + b^2$. Hence the law of cosines reduces to the theorem of Pythagoras if the triangle is a right triangle.

46. Applications of the Law of Cosines. The law of cosines may be used to calculate the remaining parts of any oblique triangle in which we have given two sides and their included angle (Case III) or all three sides (Case IV). The following examples will illustrate these two cases.

EXAMPLE 1. To compute the distance across a barrier from point A to point B, a point C is taken. The distances CA, CB and the angle ACB are found to be 30 yards, 50 yards, and 75° 15′ respectively. Find the distance AB.

SOLUTION. Using relation (6), § 45, we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Substituting, we obtain

$$c^2 = 50^2 + 30^2 - 2(50)(30)(0.2546),$$
$$c^2 = 2636.2,$$
$$c = 51.34 \text{ yards}.$$

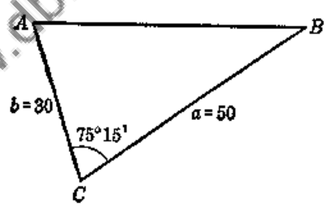


FIG. 76

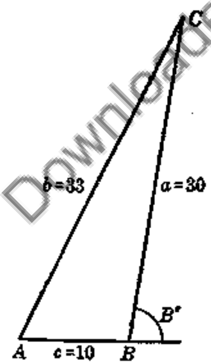


FIG. 77

EXAMPLE 2. The foot of a ladder 33 feet long is 10 feet from the base of a buttress. Find the inclination of the face of the buttress if the ladder reaches a point 30 feet up its face.

SOLUTION. Let B' be the inclination of the face of the buttress. We can find the angle B from (5), § 45. Thus

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

Substituting, we have

$$\cos B = \frac{30^2 + 10^2 - 33^2}{(2)(10)(30)} = -0.1483$$

Then

$$B = 98° 32', \quad \text{and} \quad B' = 81° 28'.$$

EXAMPLE 3. Two forces of magnitudes 30 and 60 pounds make an angle of 65° with each other. Find the magnitude of the resultant force.

SOLUTION. In mechanics, we learn that if two forces are applied at a point O, (Fig. 78) acting in the directions OA and OB, and if the lengths of OA and OB are equal to the magnitudes of these forces, then the diagonal, OC, of the parallelogram $OACB$ represents in both magnitude and direction, the ***resultant*** force. The forces represented by OB and OA are called the ***components*** of the force represented by OC. It is sometimes convenient to speak of the force OC as being resolved into two components, OA and OB. The physical quantities, velocities and accelerations may be similarly represented in length and direction. Quantities such as forces, velocities, and accelerations which have both magnitude and direction are called ***vectors.***

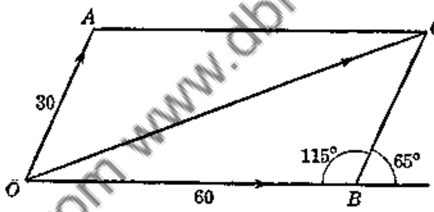


FIG. 78

Applying the law of cosines, we have

$$\begin{aligned}\overline{OC}^2 &= \overline{OB}^2 + \overline{BC}^2 - 2OB\cdot OA\cdot\cos OBC,\\ &= (60)^2 + (30)^2 - (2)(60)(30)(\cos 115°)\\ &= 3600 + 900 - (3600)(-0.4226)\\ &= 6021.\end{aligned}$$

Hence $OC = 77.60$ pounds.

EXERCISES

1. Find the angles of the triangle whose sides are 15, 17, and 19.

2. The resultant of two forces of 30 and 60 pounds is 77.6 pounds. The angle between the two forces is 65°. Use the law of sines to find the angles between the resultant and the two components. See example three above.

3. The inclination of a buttress is 85° 28′. A ladder, whose foot is 10 feet from the base of the buttress, reaches 30 feet up its face. Find the length of the ladder and its inclination.

4. In a triangle in which $a = 9$, $b = 12$, C takes the following values: 30°, 45°, 60°, 90°, 120°, 135°, and 150°, find c in each case.

5. Find the area of a square whose perimeter is equal to the perimeter of the triangle in which $a = 30$, $b = 40$, $C = 32°$.

6. The sides of a triangle are, $a = 35$, $b = 40$, $c = 31$. Extend BC by 50 units to D. Find the angle at A subtended by CD.

7. On the side b of the triangle ABC in Exercise 6 an equilateral triangle ACD is constructed, exterior to the triangle ABC. Find BD.

8. Construct a square on the side b (Exercise 6) exterior to the triangle ABC. Find the distances of its vertices from B.

9. A ship steams N 35° E from a point A at a rate of 18 miles per hour. After 1.5 hours it alters its course to N 10° W. Find its distance from A at the end of 4 hours.

10. At a point O, two forces, $F_1 = 40$ lbs., $F_2 = 50$ lbs., act in the directions N 47° 17′ E and N 25° W, respectively. Find the magnitude of the resultant force and its direction.

11. A force of 80 pounds is resolved into two components which make angles of 40° and 55° with the direction of the original force. Find the magnitude of each component.

12. In any triangle ABC show that

$$a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C).$$

13. In the triangle ABC, construct the altitudes AD, BE, and CF. Show by the law of cosines that $EF = a \cos A$, $ED = c \cos C$, and $DF = b \cos B$.

14. On the side a of any triangle ABC, an equilateral triangle BCA' is constructed exterior to the triangle ABC. In like manner construct equilateral triangles on the sides b and c, exterior to the triangle. Prove by the law of cosines that $AA' = BB' = CC'$.

15. In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{(a^2 + b^2 + c^2)}{2abc}.$$

16. Given A, a, and c in the formula for the law of cosines, show that

$$b = c \cos A \pm \sqrt{a^2 - c^2 \sin^2 A}.$$

Discuss and interpret the cases, $a = c$, $a > c \sin A$, $a = c \sin A$.

17. A tree, 60 feet high, standing on the side of a hill which is inclined at a constant angle of 15° to the horizontal is broken over by a storm. Its top strikes the ground 20 feet down the hill from the base of the tree. Find the length of the part broken over.

18. If the top of the tree in Exercise 17 struck the ground 20 feet up the hill from the base of the tree, find the length of the part broken over.

19. In Exercise 17, if the part of the tree broken over forms the base of an isosceles triangle, find the length of that part of the tree which remains standing.

47. Area. *In any triangle ABC, the area is*

$$S = \tfrac{1}{2}ab \sin C.$$

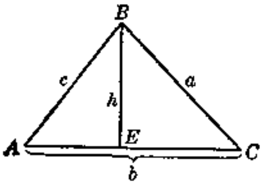


FIG. 79

PROOF. Draw the altitude, $BE = h$, in triangle ABC (Fig. 79).

Then we have

$$S = \tfrac{1}{2}bh, \quad \text{and} \quad h = a \sin C.$$

Therefore

(7) $$S = \tfrac{1}{2}ab \sin C.$$

Since a and b are any two sides, we have also

(8) $$S = \tfrac{1}{2}bc \sin A,$$

(9) $$S = \tfrac{1}{2}ac \sin B.$$

The relations (7), (8), and (9) may be expressed as follows: *The area of any triangle is equal to one-half the product of any two sides multiplied by the sine of their included angle.*

In a triangle in which a side and two angles are given, its area may be found by replacing b by its equal, $c \sin B/\sin C$, in (8). There results

(10) $$S = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}.$$

Similar expressions are obtained if the given side is a or b.

In case three sides of a triangle are given, the area may be calculated by the formula

(11) $$S = \sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a + b + c$. To obtain (11), we use (7). Squaring both sides of (7), we have

$$4S^2 = a^2b^2 \sin^2 C = a^2b^2(1 - \cos^2 C)$$
$$= a^2b^2(1 + \cos C)(1 - \cos C).$$

By replacing $\cos C$ by $\dfrac{a^2 + b^2 - c^2}{2ab}$, by § 45, Exercise 7, we have

$$4S^2 = a^2b^2\left(1 + \frac{a^2 + b^2 - c^2}{2ab}\right)\left(1 - \frac{a^2 + b^2 - c^2}{2ab}\right),$$

$$16S^2 = (2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2),$$
$$16S^2 = [(a + b)^2 - c^2][c^2 - (a - b)^2],$$
$$16S^2 = (a + b + c)(a + b - c)(-a + b + c)(a - b + c).$$

Extracting the square root, we have

$$4S = \sqrt{(a + b + c)(a + b - c)(a - b + c)(-a + b + c)}.$$

Introducing a symbol s for half the perimeter of the triangle, that is, $a + b + c = 2s$, we have

$$a + b - c = (a + b + c) - 2c = 2s - 2c = 2(s - c),$$
$$a - b + c = (a + b + c) - 2b = 2s - 2b = 2(s - b),$$
$$-a + b + c = (a + b + c) - 2a = 2s - 2a = 2(s - a).$$

Substituting in the above formula, we have formula (11), which is often called ***Heron's formula.***

EXERCISES

Find the area of each of the following triangles, given:

1. $a = 25$, $b = 30$, $C = 75° 32'$.
2. $b = 13$, $c = 21$, $A = 95° 13'$.
3. $c = 50$, $a = 75$, $B = 17° 48'$.
4. $a = 50$, $B = 35°$, $C = 48° 23'$.
5. $b = 100$, $C = 71° 18'$, $A = 65° 13'$.
6. $c = 120$, $A = 75°$, $B = 85° 18'$.
7. $a = 13$, $b = 15$, $c = 17$.
8. $a = 20$, $b = 30$, $c = 45$.
9. $a = 100$, $b = 120$, $c = 130$.

10. Find S for an equilateral triangle of side a using (7), (10), and (11).

11. Show that $S = rs$, where S is the area of ABC (Fig. 80), r is the radius of the inscribed circle, and where $2s = a + b + c$.

HINT. Area ABC = area AOB + area BOC + area COA.

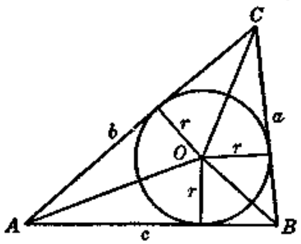


FIG. 80

12. Using (11), § 47, and the formula of Exercise 11, show that

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

13. A gardener has a triangular plot of ground whose sides are 30, 40, and 60 feet. Find the area of the largest circular flower bed the gardener can have. Use $\pi = 3.142$

14. Prove that $S = 2R^2 \sin A \sin B \sin C$, where S is the area of any triangle ABC and where R is the radius of the circumscribed circle of the triangle ABC.

HINT. Use Exercise 7, § 44, and (7), § 47.

15. In any triangle, show that $S = abc/(4R)$.

16. Using Exercise 11, show that $S = Rr(\sin A + \sin B + \sin C)$.

17. Find r given the data in Exercises 7, 8, 9. Use Exercise 12.

18. Find R given the data in Exercises 4, 5, 6. Use Exercise 7, § 44.

19. Find the area of the circumcircle of the triangle whose sides are 15, 20, 25. Use $\pi = 3.142$

20. Find the ratio of the areas of the incircle and circumcircle of the triangle whose sides are 11, 13, 16.

CHAPTER VII

ADDITION THEOREMS AND RELATED FORMULAS

48. Formulas for sin $(A + B)$ and cos $(A + B)$. In deriving formulas especially adapted to the solution of oblique triangles by logarithms (Chapter IX), and in many applications of trigonometry to other subjects, frequent use is made of formulas for expressing trigonometric functions of the sum or difference of two angles in terms of functions of the separate angles.

We shall now establish* two such formulas:

(1) $$\sin (A + B) = \sin A \cos B + \cos A \sin B,$$

(2) $$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

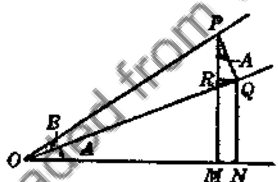


FIG. 81

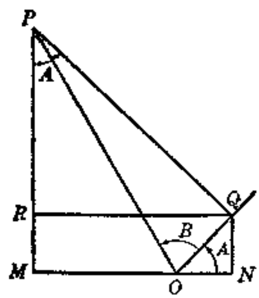


FIG. 82

For the present we shall assume that A and B are each positive acute angles. Two cases arise: $A + B$ acute (Fig. 81), and $A + B$ obtuse (Fig. 82). In each of these figures, $A = NOQ$ and $B = QOP$. Then $A + B = NOP$.

In the figures, the line PQ is drawn perpendicular to OQ. The lines MP and NQ are perpendicular to ON or ON produced, and QR is parallel to ON. Then, from the figure, by the definitions of sines and cosines of A and B, we have from the right triangles ONQ, OQP, and PRQ,

* Suggestions leading to other proofs of these important theorems are given in Exercises 57–59 at the end of the chapter.

$$\sin A \cos B + \cos A \sin B = \frac{NQ}{OQ} \cdot \frac{OQ}{OP} + \frac{RP}{PQ} \cdot \frac{PQ}{OP}$$
$$= \frac{NQ}{OP} + \frac{RP}{OP} = \frac{MR}{OP} + \frac{RP}{OP}$$
$$= \frac{MP}{OP} = \sin (A + B),$$

by the definition of the sine.

Hence, stated in words, *the sine of the sum of two angles is equal to the product of the sine of the first and the cosine of the second plus the product of the cosine of the first and the sine of the second.*

This statement is called the ***addition theorem of the sine,*** and formula (1) is called the ***addition formula of the sine.***

To guard against the false inference that $\sin (A + B)$ is the sum of $\sin A$ and $\sin B$, the student should verify that $\sin 90°$ is less than the sum of $\sin 30°$ and $\sin 60°$.

Similarly, we observe from the figure that

$$\cos A \cos B - \sin A \sin B = \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{RQ}{PQ} \cdot \frac{PQ}{OP}$$
$$= \frac{ON}{OP} - \frac{RQ}{OP} = \frac{ON}{OP} - \frac{MN}{OP}$$
$$= \frac{OM}{OP} = \cos (A + B).$$

Hence, the *cosine of the sum of two angles is equal to the product of the cosine of the first and the cosine of the second minus the product of the sine of the first and sine of the second.*

This is called the ***addition theorem of the cosine,*** and (2) is the ***addition formula of the cosine.***

49. Extension of the Formulas for $\sin (A + B)$ and $\cos (A + B)$ to All Angles without Restriction. In § 48, we have proved the addition formulas when A and B are positive acute angles. It may be shown that these formulas hold for all angles by showing that they continue to hold if either A or B is increased by 90°. It seems that the permanence of form of formulas (1) and (2) when 90° is added to an angle may well be shown by exercises.

EXAMPLE 1. Given

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$
$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

where α and β are any acute angles, and $\gamma = \beta + 90°$; show that

(3) $$\sin(\alpha + \gamma) = \sin\alpha\cos\gamma + \cos\alpha\sin\gamma.$$

SOLUTION. Since $\gamma = \beta + 90°$,

(a) $$\begin{aligned}\sin(\alpha+\gamma) &= \sin(\alpha+\beta+90°) = \cos(\alpha+\beta) \quad (\S\,30)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

But

$$\cos\beta = \cos(\gamma - 90°) = \sin\gamma \qquad (\S\,30)$$

and

$$\sin\beta = \sin(\gamma - 90°) = -\cos\gamma \qquad (\S\,30).$$

Hence, by substitution in (a), we have

$$\sin(\alpha+\gamma) = \sin\alpha\cos\gamma + \cos\alpha\sin\gamma,$$

which was to be proved.

Note that, since α and β enter symmetrically in the formula for $\sin(\alpha + \beta)$, we could have increased α by 90° as well as β.

Similarly, it may be shown that the permanence of form of formula (1) would remain if either α or β were decreased by 90°.

EXAMPLE 2. Given

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\end{aligned}$$

and $\gamma = \beta \pm 90°$, show that

(4) $$\cos(\alpha+\gamma) = \cos\alpha\cos\gamma - \sin\alpha\sin\gamma.$$

From the results of Examples 1 and 2, it follows that if the formulas (1) and (2) are true for given angles α and β, they remain true for angles obtained by increasing or decreasing either of the given angles by 90°. Since any angle may be obtained from an acute angle by repeated additions or subtractions of 90°, the addition formulas (1) and (2) are true for all angles.

50. Formulas for $\tan(A + B)$ and $\cot(A + B)$. By (7), § 35,

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Dividing both numerator and denominator by $\cos A\cos B$, we have

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}},$$

that is,

(5) $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Express this formula in words.

Similarly, we have

$$\cot(A+B) = \frac{\cos(A+B)}{\sin(A+B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B},$$

or, dividing numerator and denominator by $\sin A \sin B$,

(6) $$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

Formulas (5) and (6) are the ***addition formulas*** for the tangent and cotangent respectively.

51. Functions of $(A - B)$. By the use of one or more of the formulas (1), (2), (5), (6), and the facts that $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$, we obtain the following ***subtraction formula:***

$$\begin{aligned}\sin(A-B) &= \sin[A + (-B)] \\ &= \sin A \cos(-B) + \cos A \sin(-B).\end{aligned}$$

Then

(7) $$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Similarly,

(8) $$\cos(A-B) = \cos A \cos B + \sin A \sin B,$$

(9) $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

(10) $$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

EXERCISES

In the following exercises, it is assumed that the student knows the values of the functions of 0°, 30°, 45°, 60°, and 90°.

1. Find $\sin 75°$ in terms of radicals.

SOLUTION. Note that $75° = 45° + 30°$; hence

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2}\cdot\frac{\sqrt{3}}{2} + \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).$$

2. Show that $\cos(60° + 30°) < \cos 60° + \cos 30°$.
3. Show that $\cos 75° = \frac{1}{4}(\sqrt{6} - \sqrt{2}) = \sin 15°$.
4. Show that $\tan 75° = 2 + \sqrt{3} = \cot 15°$.
5. Show that $\sin 15° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$ by using the fact that $15° = 45° - 30°$.
6. Show that $\tan 15° = 2 - \sqrt{3}$ by using the fact that $15° = 45° - 30°$.
7. Given $\tan x = 3/4$, $\sin y = 5/13$, and x and y both positive acute angles. Find the values of $\tan(x + y)$, $\sin(x + y)$, $\cos(x + y)$, $\sin(x - y)$.
8. Given $\tan x = 4/3$, $\tan y = 3/4$ where x and y are both positive acute angles. Find the values of $\sin(x - y)$, $\cos(x - y)$, and $\tan(x - y)$.
9. If $\cos A = 1/3$ and $\cos B = 2/3$, find $\cos(A + B)$ and $\cos(A - B)$ if A and B are acute angles.
10. If $\cos A = 1/3$, and $\sin B = 3/4$, find $\cos(A - B)$ if A and B are acute angles.

Show whether the statement in each of the Exercises 11–22 is an identity.

11. $\sin(45° + \theta) = \frac{1}{2}\sqrt{2}(\cos\theta + \sin\theta)$.
12. $\sin(30° + x) + \cos(60° + x) = \cos x$.
13. $\cos(30° - x) - \cos(30° + x) = \sin x$.
14. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.
15. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$.
16. $\tan(45° + A) + \cot(A - 45°) = 0$.
17. $\tan(45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$. **18.** $\tan(45° - \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$.
19. $\cos(60° + x) + \sin(30° + x) = \cos x$.
20. $\sin(60° + x) = \frac{1}{2}(\sqrt{3}\cos x + \sin x)$.
21. $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.
22. $\tan(A + 45°)\tan(A - 45°) = -1$.
23. Given $\sin(A + B) = 1$, $\sin A = 3/5$, and that A and B are positive acute. Find $\sin B$, $\cos B$, and $\tan B$.
24. Given that $\sin(A + B) = 56/65$, $\sin B = 12/13$, and that A and B are positive acute. Find $\sin A$, $\cos A$, and $\tan A$.
25. Find $\sin 150°$ from functions of 180° and 30°.
26. Find $\cos 105°$ from the functions of 45° and 60°.
27. If $\sin x = 0.6$ and $\tan y = 1$, find the values of $\sin(x - y)$, $\tan(x - y)$.
28. If $\tan x = 0.5$ and $\cot y = 0.5$, show that $\cos(x + y) = 0$.
29. If $\tan A = 1$, and $\tan(A + B) = -2 - \sqrt{3}$, find $\tan B$.
30. If $\tan A = 4$, $A + B = 45°$, find $\tan B$.
31. If $\sin x = 0.309$ and $\cos y = 0.809$ when x and y are acute angles, find $\sin(x + y)$ and verify the result by using the table on pp. 21–26.
32. Work Exercise 31 if x is in the second quadrant.
33. Find $\cos 120°$ by using (a) $120° = 60° + 60°$, (b) $120° = 90° + 30°$.
34. Find $\sin 135°$ by using $135° = 90° + 45°$.

35. Find tan 210° by using $210° = 180° + 30°$.
36. Find cos 300° by using $300° = 360° - 60°$.

Derive the following formulas from the addition and subtraction formulas:

37. $\sin x = \cos(x - 90°)$.
38. $\cos(180° + x) = -\cos x$.
39. $\sin(180° - x) = \sin x$.
40. $\cos(270° - x) = -\sin x$.
41. $\sin(270° - x) = -\cos x$.
42. $\cos(360° - x) = \cos x$.
43. Derive the formula

$$\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z.$$

Hint. Write $\sin(x + y + z) = \sin[x + (y + z)]$ and apply (1), § 48.

44. Derive the formula

$$\cos(x + y + z) = \cos x \cos y \cos z - \cos x \sin y \sin z - \cos y \sin x \sin z - \cos z \sin x \sin y.$$

45. Derive the formula

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}.$$

52. Functions of Double Angles. In the addition formulas

(1) $$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

(2) $$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

(5) $$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

let $B = A$, then $(A + B)$ becomes the *double angle* $2A$, and after slight reductions, we obtain

(11) $$\sin 2A = 2 \sin A \cos A,$$

(12) $$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1,$$

(13) $$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

The significance of formulas (11), (12), and (13) may well be emphasized by expressing them in words in each of two ways. Thus, (11) may be stated in two ways as follows:

1. *The sine of twice an angle is twice the product of the sine and cosine of the angle.*

2. *The sine of an angle is equal to twice the product of the sine and cosine of an angle half as large as the given angle.*

ORAL EXERCISES

1. Express sin 60° in terms of functions of 30°.
2. Express sin 61° in terms of functions of half as large an angle.
3. Express each formula (12) in words in two ways.
4. Express formula (13) in words in two ways.
5. Given $\sin x = 3/5$, find $\sin 2x$.
6. Given $\sin (3x/2) = 7/25$, find $\sin 3x$.
7. Given $\tan (3x/2) = 12/5$, find $\tan 3x$.

53. Functions of Half Angles. If in formula (12) of § 52, we regard $2A$ as the angle first to be considered, then the angles in the right members are half angles. To put this point more clearly in evidence, let $A = x/2$. Then, from (12), we have

$$\cos x = 1 - 2\sin^2\frac{x}{2}, \quad \text{and} \quad \cos x = 2\cos^2\frac{x}{2} - 1.$$

If we solve the first of these equations for $2\sin^2(x/2)$ and the second for $2\cos^2(x/2)$, we find

$$2\sin^2\frac{x}{2} = 1 - \cos x, \quad \text{and} \quad 2\cos^2\frac{x}{2} = 1 + \cos x,$$

whence

(14) $$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}},$$

(15) $$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}.$$

The appropriate signs in the right members are determined by the quadrant of the angle $x/2$. If x is a positive angle not greater than 180°, $x/2$ is in the first quadrant and the sign + must be chosen in both (14) and (15). From (14) and (15), we find

(16) $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}},$$

the appropriate sign being again determined by the quadrant of the angle $x/2$. By multiplying numerator and denominator of the right member of (16) by $\pm\sqrt{1+\cos x}$, we get

$$\tan\frac{x}{2} = \frac{\sqrt{1-\cos^2 x}}{1+\cos x} = \frac{\sin x}{1+\cos x}.$$

Similarly, by multiplying numerator and denominator of (16) by $\pm\sqrt{1-\cos x}$, we obtain

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sqrt{1-\cos^2 x}} = \frac{1-\cos x}{\sin x}.$$

Collecting our results, we may write

$$\textbf{(17)}\qquad \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}.$$

Formulas (14), (15), (16), (17) express the functions of an angle in terms of functions of an angle twice as large as the given angle.

EXERCISES

1. If $\sin A = 4/5$, find $\sin 2A$; also $\cos 2A$.
2. If $\cos A = 12/13$, find $\sin 2A$; also $\tan 2A$.
3. If $\sin (x/2) = 0.6$, find $\sin x$; also $\tan x$.
4. If $\cos (x/2) = 0.3$, find $\sin x$; also $\tan x$.
5. If $\csc x = 5/4$, find $\cos 2x$.

Assuming that the values of functions of 30°, 45°, and 60° are known, find the value of each of the following functions without using tables.

6. $\sin 120°$ from functions of 60°
7. $\sin 90°$ from functions of 45°
8. $\tan 120°$ from $\tan 60°$
9. $\cos 135°$ from $\cos 45°$
10. $\sin 15°$ from $\cos 30°$
11. $\cos 15°$ from $\cos 30°$
12. $\tan 15°$ from $\sin 30°$ and $\cos 30°$
13. $\sin 22\frac{1}{2}°$ from $\cos 45°$
14. $\cos 22\frac{1}{2}°$ from $\cos 45°$
15. $\tan 22\frac{1}{2}°$ from $\sin 45°$ and $\cos 45°$

Express orally each of the following in terms of functions of half the angle.

16. $\sin 72°$
17. $\sin 8x$
18. $\cos 10x$
19. $\tan 4x$
20. $\sin 8°$
21. $\sin 3\theta$
22. $\cos \frac{1}{2}\theta$
23. $\sin 120°$
24. $\cos 300°$
25. $\sin (2x + 60°)$
26. Express $\sin 3\theta$ in terms of $\sin \theta$.

HINT. In (1), § 48, make $A = 2\theta$ and $B = \theta$. Use (11) and (12), § 52.

27. Express $\cos 3\theta$ in terms of $\cos \theta$.
28. Express $\tan 3\theta$ in terms of $\tan \theta$.
29. Derive the formula $\sin 4\theta = 4 \cos^3\theta \sin\theta - 4 \cos\theta \sin^3\theta$.

HINT. Apply (11) and (12), § 52, twice.

30. Derive the formula $\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$.
31. Given $\cos\theta = -1/2$ where θ is in the second quadrant; find the values of $\sin(\theta/2)$, $\cos(\theta/2)$.

Express each of the following in terms of functions of an angle twice as large as the given angle.

32. $\cos^2 2x$
33. $\sin^2 3x$
34. $\tan 2x$
35. $\sin^2 20°$

Prove the following identities:

36. $2 \csc 2x = \sec x \csc x$.
37. $\cos^4 x - \sin^4 x = \cos 2x$.

38. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$.

39. $\cot \dfrac{x}{2} = \dfrac{\sin x}{1 - \cos x}$.

40. $(\sin x - \cos x)^2 + \sin 2x = 1$.

41. $(\sin x + \cos x)^2 = 1 + \sin 2x$.

42. $\sin x \left(\tan \dfrac{x}{2} + \cot \dfrac{x}{2}\right) = 2$.

43. $\tan \dfrac{x}{2} = \cot \dfrac{x}{2} - 2 \cot x$.

44. $\tan (45° + x) - \tan (45° - x) = 2 \tan 2x$.

45. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

46. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$.

47. $\dfrac{\cos 3x}{\sin x} + \dfrac{\sin 3x}{\cos x} = 2 \cot 2x$.

48. $\cos x \cos 3x - \sin x \sin 3x = \cos 4x$.

49. $\sec 2x + \tan 2x = \tan (45° + x)$.

50. If $\cos 2x = -\frac{1}{2}$ and x is in the second quadrant, find the values of $\cos (x/2)$, $\sin (x/2)$, and $\tan (x/2)$.

Simplify each of the following expressions.

51. $2 \sin^3 x \cos x + 2 \sin x \cos^3 x$

52. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$

53. $\tan x(1 + \sec 2x)$

54. $\sin 5x \cos 2x - \cos 5x \sin 2x$

55. $\csc x - \cot x$

56. $\dfrac{1 - \cos 2x}{\sin 2x}$

54. To Express the Product of Two Functions as the Sum or Difference of Functions. From the formulas

(1) $\sin (A + B) = \sin A \cos B + \cos A \sin B$,

(2) $\cos (A + B) = \cos A \cos B - \sin A \sin B$,

(7) $\sin (A - B) = \sin A \cos B - \cos A \sin B$,

(8) $\cos (A - B) = \cos A \cos B + \sin A \sin B$,

we obtain, by addition and subtraction of members,

(a) $\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$,

(b) $\sin (A + B) - \sin (A - B) = 2 \cos A \sin B$,

(c) $\cos (A + B) + \cos (A - B) = 2 \cos A \cos B$,

(d) $\cos (A + B) - \cos (A - B) = -2 \sin A \sin B$.

By solving for $\sin A \sin B$, $\cos A \cos B$, and $\sin A \cos B$ in the second members of these identities, we get the product formulas

(18) $\sin A \sin B = \frac{1}{2}[\cos (A - B) - \cos (A + B)]$,

(19) $\cos A \cos B = \frac{1}{2}[\cos (A + B) + \cos (A - B)]$,

(20) $\sin A \cos B = \frac{1}{2}[\sin (A + B) + \sin (A - B)]$.

It is a good exercise for the student to read these formulas in words. Thus, (18) reads: *The product of the sines of two angles is equal to one-half the quantity formed by subtracting the cosine of the sum of the angles from the cosine of their difference.*

EXAMPLE 1. Read formulas (19) and (20) in words.

EXAMPLE 2. Check formulas (18), (19), and (20) for $A = 60°$, $B = 30°$. The above formulas are important in integral calculus.

55. To Express the Sum or Difference of Two Like Functions as a Product of Two Functions. To derive formulas for the sum or difference of sines or cosines of two angles, we need only make

$$A + B = x, \tag{e}$$

$$A - B = y \tag{f}$$

in (a), (b), (c), (d) of § 54. From (e) and (f)

$$A = \tfrac{1}{2}(x + y),$$

$$B = \tfrac{1}{2}(x - y).$$

Using these values in (a), (b), (c), (d), we obtain

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y), \tag{21}$$

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y), \tag{22}$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y), \tag{23}$$

$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y). \tag{24}$$

Here again the student should read the formulas in words. Thus, (21) reads: *The sum of the sines of two angles is equal to twice the product of the sine of half the sum of the angles and the cosine of half their difference.* These formulas are useful for purposes of simplification (see § 78) and in particular for putting certain expressions into suitable form for logarithmic computation.

EXAMPLE 1. Show that $\sin 40° + \sin 20° = \cos 10°$.

SOLUTION. From (21),

$$\sin 40° + \sin 20° = 2 \sin 30° \cos 10° = \cos 10°,$$

since $\sin 30° = \frac{1}{2}$.

EXAMPLE 2. Show that

$$\frac{\sin 5\theta - \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot 4\theta.$$

SOLUTION. Applying (22) and (24) to the numerator and denominator of the left member, taking $x = 5\theta$, $y = 3\theta$, we obtain

$$\frac{\sin 5\theta - \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \frac{2 \cos \frac{1}{2}(5\theta + 3\theta) \sin \frac{1}{2}(5\theta - 3\theta)}{2 \sin \frac{1}{2}(5\theta + 3\theta) \sin \frac{1}{2}(5\theta - 3\theta)}$$

$$= \frac{2 \cos 4\theta \sin \theta}{2 \sin 4\theta \sin \theta} = \cot 4\theta.$$

EXERCISES

Express as the sum or difference of two functions:

1. $2 \sin 3A \cos A$
2. $2 \cos 4A \cos 2A$
3. $2 \cos 5x \sin 3x$
4. $2 \sin 3x \sin 2x$
5. $2 \sin 7\theta \sin 3\theta$
6. $2 \sin (5\theta/2) \cos (\theta/2)$
7. $\cos (A + B) \cos (A - B)$
8. $\cos 3x \cos x$

Express as the product of two functions and simplify:

9. $\sin 75° + \sin 15°$
10. $\cos 75° + \cos 15°$
11. $\sin 75° - \sin 15°$
12. $\cos 15° - \cos 75°$
13. $\cos 42° - \cos 48°$
14. $\sin 37° + \sin 63°$
15. $\sin 50° + \cos 20°$ HINT. $\cos 20° = \sin 70°$.
16. $\cos 50° + \sin 20°$
17. $\sin 65° - \cos 35°$
18. $\cos 35° - \cos 65°$
19. $\cos 20° - \sin 50°$
20. $\sin 75° - \cos 75°$
21. $\cos 75° - \sin 75°$
22. $\cos 10° - \sin 10°$

Express as the product of two functions:

23. $\sin 5\theta + \sin \theta$
24. $\cos 5\theta + \cos 3\theta$
25. $\sin 5x - \sin x$
26. $\cos 6x - \cos 2x$
27. $\sin (A + B) + \sin (A - B)$
28. $\cos (5\theta - 2A) + \cos (5\theta + 2A)$
29. $\sin 5x - \cos (90° - x)$
30. $\sin 4x + \cos x$

Prove the identities in Exercises 31–40:

31. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan (A + B)}{\tan (A - B)}$.

32. $\dfrac{\cos x - \cos y}{\cos x + \cos y} = -\tan \dfrac{x - y}{2} \tan \dfrac{x + y}{2}$.

33. $\dfrac{\sin 3A + \sin A}{\sin 6A - \sin 2A} = \cos A \sec 4A$.

34. $\dfrac{\sin 5x - \sin x}{\cos 5x + \cos x} = \tan 2x$.

35. $\sin (x + 60°) + \sin (x - 60°) = \sin x$.

36. $\dfrac{\sin (2x - 3y) + \sin 3y}{\cos (2x - 3y) + \cos 3y} = \tan x$.

37. $\dfrac{\cos 3x + \cos x}{\sin 3x + \sin x} = \cot 2x$.

38. $\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) + \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B) = \sin A$.

39. $\cos (A + 60°) + \cos (60° - A) = \cos A$.

40. $\dfrac{\cos (n - 2)x - \cos (nx)}{\sin (n - 2)x + \sin (nx)} = \tan x$.

Show whether each of the equalities 41–52 is an identity:

41. $2 \sin (x + 45°) \sin (x - 45°) = -\cos 2x.$

42. $\dfrac{\cos 3x - \cos 4x}{\sin 4x + \sin 3x} = \tan \frac{1}{2}x.$ **43.** $\dfrac{\sin 50° - \sin 10°}{\cos 50° - \cos 10°} = -\sqrt{3}.$

44. $\cos 2x - \cos 6x = 16 \sin^2 x \cos^2 x(1 - 2 \sin^2 x).$

45. $\dfrac{\cos x + \cos 2x + 1}{\sin x + \sin 2x} = \cot x.$ **46.** $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x.$

47. $\dfrac{\sin x - \cos x + 1}{\sin x + \cos x + 1} = \tan \dfrac{x}{2}\cdot$

48. $\sin 2A + \sin 4A + \sin 6A = 4 \cos A \cos 2A \sin 3A.$

49. $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A.$

50. $\sin A + \sin B + \sin (A + B) = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{A+B}{2}\cdot$

51. $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \dfrac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}\cdot$

52. $\dfrac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x.$

53. When $A + B + C = 180°$, prove the formula

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}\cdot$$

SOLUTION.

$$(\sin A + \sin B) + \sin C = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) + \sin C$$

$$= 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) + 2 \sin \frac{C}{2} \cos \frac{C}{2}\cdot$$

But

$$\sin \tfrac{1}{2}(A + B) = \cos \frac{C}{2} \text{ since } \frac{A+B}{2} + \frac{C}{2} = 90°.$$

Hence

$$\sin A + \sin B + \sin C = 2 \cos \frac{C}{2}\left[\cos \tfrac{1}{2}(A - B) + \sin \frac{C}{2}\right]$$

$$= 2 \cos \frac{C}{2}\left(\cos \frac{A-B}{2} + \cos \frac{A+B}{2}\right).$$

$$= 2 \cos \frac{C}{2}\left(2 \cos \frac{A}{2} \cos \frac{B}{2}\right) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}\cdot$$

54. $\sin A + \sin B - \sin C = 4 \sin (A/2) \sin (B/2) \cos (C/2).$

55. $\cos A + \cos B + \cos C = 1 + 4 \sin (A/2) \sin (B/2) \sin (C/2).$

56. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

HINT. $\tan C = -\tan (A + B).$

57. Using A and B as two angles of a triangle (Fig. 83), prove the addition theorem

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

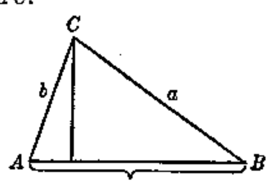


FIG. 83

SUGGESTIONS. From what was shown in Exercise 7, § 44, it follows that in any triangle

(a) $a = 2R \sin A, \qquad b = 2R \sin B, \qquad c = 2R \sin C,$

where R is the radius of the circle circumscribing the triangle ABC.

Note from Fig. 83 that

(b) $$c = a \cos B + b \cos A.$$

From (a) and (b), we get

(c) $$2R \sin C = 2R \sin A \cos B + 2R \sin B \cos A.$$

Complete the proof by dividing both members of (c) by $2R$, and then by using the fact that $C = 180° - (A + B)$.

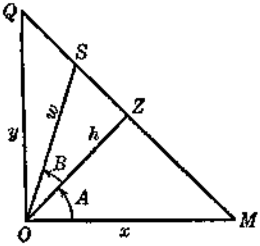


FIG. 84

58. Establish the addition theorem

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

by equating the area of triangle OMS, Fig. 84, to the sum of the areas of triangles OMZ and OZS.

SUGGESTION. From Fig. 84, by using (7), § 47,

$$\tfrac{1}{2}wx \sin (A + B) = \tfrac{1}{2}hx \sin A + \tfrac{1}{2}hw \sin B.$$

Divide through by $\tfrac{1}{2}wx$, and

$$\sin (A + B) = \frac{h}{w} \sin A + \frac{h}{x} \sin B.$$

Complete the proof by replacing h/w and h/x by using the definition of the cosine.

59. Establish the addition theorem

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

by starting with the relation

$$\text{area } OSQ = \text{area } OZQ - \text{area } OZS.$$

SUGGESTION. From Fig. 84, and (7), § 47,

$$\tfrac{1}{2}wy \sin (90° - A - B) = \tfrac{1}{2}hy \sin (90° - A) - \tfrac{1}{2}hw \sin B.$$

Divide through by $\tfrac{1}{2}wy$ and complete the solution.

60. From the right triangles AMP and MPB (Fig. 85) derive the following formulas:

$$\sin 2x = 2 \sin x \cos x,$$
$$2 \cos^2 x = 1 + \cos 2x,$$
$$2 \sin^2 x = 1 - \cos 2x.$$

$$\tan x = \frac{1 - \cos 2x}{\sin 2x} = \frac{\sin 2x}{1 + \cos 2x}.$$

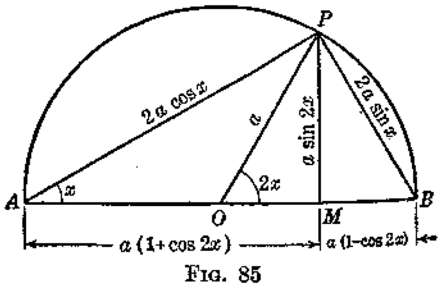


FIG. 85

CHAPTER VIII

LOGARITHMS

56. Introduction. The amount of numerical work involved in performing the operations of multiplication, division, raising to a power, and extracting a root may be greatly reduced by the use of logarithms. In the applications of trigonometry, which usually involve the solution of triangles, multiplications and divisions occur very frequently (see Chapter IX). By the use of logarithms such operations as are indicated by

$$(13.7)^{1.41}, \qquad (1.007)^{3/8}, \qquad \sqrt[3]{92.7}, \quad \sqrt[7]{0.0071}$$

can be carried out very efficiently.

57. Logarithms. *A logarithm is an exponent.* Consider the following powers of 10:

$10^{-4} = 0.0001$	$10^1 = 10$
$10^{-3} = 0.001$	$10^2 = 100$
$10^{-2} = 0.01$	$10^3 = 1000$
$10^{-1} = 0.1$	$10^4 = 10000$
$10^0 = 1$	. . . .

The exponent of 10 in the left member of each of these equalities is called the ***logarithm to the base* 10** of the corresponding number in the right member. Thus -4 is the logarithm of 0.0001, and 4 is the logarithm of 10000.

Logarithms to the base 10 are often called *common* logarithms. It is usual to use *log* as an abbreviation for the word *logarithm.*

It is very important that the student should see clearly that $10^3 = 1000$, and $\log 1000 = 3$ are merely two different ways of stating the same thing. The former is the *exponential* form, the latter the *logarithmic* form.

EXERCISES

Change each of the following statements to logarithmic form.

1. $10^2 = 100.$
2. $10^{0.5} = 3.1623$
3. $10^{2.653} = 450.$
4. $10^{-3} = 0.001$
5. $10^{-0.432} = 0.37$
6. $10^{2/3} = 4.6416$

Change each of the following statements to exponential form.

7. $\log 1000 = 3.$
8. $\log 9.9 = 0.9956$
9. $\log 0.00001 = -5.$
10. $\log 0.012 = -1.9208$
11. $\log 2.67 = 0.4265$
12. $\log 0.267 = -0.5735$

It is readily seen that for integral powers of 10 the logarithm of a number increases as the number increases. In this book we shall restrict ourselves to logarithms of positive numbers to the base 10, and shall make the following important assumption:

The logarithm of a number increases as the number increases.

58. Characteristic and Mantissa. It is evident that most numbers are not integral powers of 10. Thus 487 is more than the second power of 10 and less than the third power of 10. Hence

$$\log 487 = 2 + \text{a decimal fraction.}$$

The integral part of the logarithm is called the ***characteristic*** and the decimal part the ***mantissa.*** For example, we find from the table that $\log 1392 = 3.14364$. The characteristic is 3, and the mantissa is 0.14364.

59. Index Laws Applied to Logarithms. From the index law

$$10^x \cdot 10^y = 10^{x+y},$$

we derive an important property of logarithms. Let

$$10^x = m, \quad \text{and} \quad 10^y = n,$$

then

$$\log m = x, \quad \text{and} \quad \log n = y,$$

and

$$mn = 10^x \cdot 10^y = 10^{x+y}.$$

Hence

$$\log mn = x + y = \log m + \log n.$$

Expressed in words, this result is as follows:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

For example, we have

$$\log 14 = \log (2 \cdot 7) = \log 2 + \log 7.$$

From the index law

$$\frac{10^x}{10^y} = 10^{x-y},$$

we derive a second important property of logarithms. With the notation above we have

$$\frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}.$$

Hence

$$\log \frac{m}{n} = x - y = \log m - \log n.$$

We may express this result in words as follows:

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

For example, we have

$$\log \frac{14}{3} = \log 14 - \log 3.$$

From the index law

$$(10^x)^r = 10^{xr},$$

we derive a third important property of logarithms. With the same notation, that is, with $m = 10^x$, we have

$$m^r = (10^x)^r = 10^{xr}.$$

Hence

$$\log m^r = xr = r \log m.$$

We may express this result in words as follows:

III. *The logarithm of the rth power of a number is r times the logarithm of the number.*

For example, we have

$$\log (13.7)^{1.41} = 1.41 \log 13.7$$

In particular, if r is the reciprocal of an integer, $1/s$, say, the "raising to the rth power" becomes the "extracting of the sth root." In words, the logarithm of the real positive sth root of a number is the logarithm of the number divided by s.

For example, we have

$$\log \sqrt[3]{92.7} = \frac{1}{3} \log 92.7$$

60. Effect on the Logarithm of a Number Produced by Multiplication or Division of the Number by 10. Using the notation of § 59, we have

$$\log m = x.$$

By property I, § 59,

$$\log 10m = \log m + \log 10 = \log m + 1 = x + 1.$$

Hence multiplication of a number by 10 increases its logarithm by 1; that is, the characteristic is increased one unit, while the mantissa remains unchanged.

By property II, § 59,

$$\log \frac{m}{10} = \log m - \log 10 = \log m - 1 = x - 1.$$

Hence division of a number by 10 decreases its logarithm by 1; that is, the characteristic is decreased one unit, while the mantissa remains unchanged.

Since multiplication or division of a number by 10 simply moves the decimal point to the right or to the left, we see that the mantissa is independent of the position of the decimal point, and depends only on the sequence of digits in the number.

61. Characteristic Law. Since $\log 1 = 0$, and $\log 10 = 1$ (§ 57), and the logarithm of a number increases as the number increases, it is clear that the logarithm of any number between 1 and 10 is 0 + a decimal fraction. For example,

$$\log 7.296 = 0.86308$$

By successive multiplications and divisions by 10, we have

$$\begin{aligned}
\log 72.96 &= 1.86308,\\
\log 729.6 &= 2.86308,\\
\log 0.7296 &= -1 + 0.86308 = 9.86308 - 10,^*\\
\log 0.07296 &= -2 + 0.86308 = 8.86308 - 10,\\
\log 0.007296 &= -3 + 0.86308 = 7.86308 - 10.
\end{aligned}$$

The law for determining the characteristic of the logarithm of a number may then be stated as follows:

To find the characteristic of the logarithm of a number which has an integral part, subtract 1 *from the number of digits in the integral part.*

To find the characteristic of the logarithm of a decimal fraction subtract from 9 *the number of ciphers between the decimal point and the first significant figure. From the number so obtained subtract* 10.

* The logarithm is written in this form for convenience in computation.

EXERCISES

Give the characteristic of the logarithm of each of the following numbers.

1. 375
2. 3075
3. 12567
4. 134.51
5. 12.678
6. 3.67809
7. 0.3627
8. 0.0765
9. 10.324
10. 0.0003678
11. 123.564
12. 100002
13. Given log 3.845 = 0.58490, find log 384.5
14. Given log 47943 = 4.68075, find log 0.0047943
15. Given log 0.067333 = 8.82823 − 10, find log 673330.
16. Find log 9134, given log 9.134 = 0.96066
17. Find log 0.0003706, given log 37.06 = 1.56891
18. Find log 7637000, given log 0.07637 = 8.88292 − 10.

Find the characteristic of the logarithm of each of the following numbers.

19. $26(10)^3$
20. 14/19
21. $26(10)^{-4}$
22. $27^{1/4}$

62. Explanation of the Logarithmic Table I. Logarithms of Numbers. On pages 1–19 we give a five-place table of the logarithms of numbers. The first column on each page has the letter N at the top and bottom. The letter N is an abbreviation for number. The other columns have at the top and bottom the numbers 0, 1, 2, 3, $\cdots$, 9.

The table contains the mantissas of all four-digit numbers, the first three digits in the N-column, the fourth digit at the top and bottom of another column. It should be noted that the first two digits of the mantissa are given only in the column headed 0. The characteristic of the logarithm can always be readily supplied when needed.

63. The Logarithm of a Given Number.

Example 1. Find the logarithm of 1397.

To find the mantissa corresponding to the number 1397, we locate the intersection of the row that has 139 in the N-column, with the column that has 7 at the top. We thus find that the mantissa is 14520. The characteristic is 3, so that

$$\log 1397 = 3.14520$$

Example 2. Find the logarithm of 785.64.

This number has more than four digits, so that the mantissa of its logarithm is not given in the table. It may, however, be approximated from mantissas in the table by the process of interpolation by proportional parts, previously explained in § 43. In the tables we find that

the mantissa of log 78560 is 89520,
the mantissa of log 78570 is 89526.

Since the number 78564 lies 4/10 of the way from 78560 to 78570, by proportional parts the mantissa of its logarithm lies 4/10 of the way from the mantissa of log 78560 to the mantissa of log 78570, that is, 4/10 of the way from 89520 to 89526.

The two mantissas differ by 6, and 4/10 of 6 is 2.4. Hence the correction 2 is to be added to the smaller mantissa 89520. Thus the mantissa of log 78564 is 89522. The characteristic of the logarithm of 785.64 is 2, so that

$$\log 785.64 = 2.89522$$

EXERCISES

Obtain from Table I the logarithm of each of the following numbers.

1. 67	2. 123	3. 3256	4. 75.61
5. 2000	6. 876.9	7. 0.56	8. 0.0785
9. 17.865	10. 278.86	11. 0.78682	12. 0.0035467
13. 6.7091	14. 23.568	15. 123.91	16. 56789

64. The Number that Corresponds to a Given Logarithm.

EXAMPLE 1. Find the number whose logarithm is 2.76103.

To find the number corresponding to the mantissa 76103, we locate the row and column in which this mantissa lies. We find that it lies in the row that has 576 in the N-column, and in the column that has 8 at the top.

The sequence of digits corresponding to the mantissa 76103 is, therefore, 5768. The characteristic is 2, so that the number whose logarithm is 2.76103 is 576.8.

EXAMPLE 2. Find the number whose logarithm is 1.68792.

The mantissa 68792 is not given in the table, but it lies between the two adjacent mantissas 68789 and 68797 of the table. The mantissa 68789 corresponds to the number 48740, and the mantissa 68797 corresponds to the number 48750.

The mantissa 68792 lies at 3/8 of the way from 68789 to 68797. Hence, by the process of interpolation by proportional parts, the number whose mantissa is 68792 is

$$48740 + \frac{3}{8} \text{ of } 10 = 48744.$$

Thus

$$\log 48.744 = 1.68792$$

EXERCISES

Obtain from the table the five digit number corresponding to each of the following logarithms.

1. 2.13577	2. 1.23578	3. 3.47885
4. 0.56789	5. 9.78522 − 10	6. 8.96699 − 10
7. 2.57000	8. 4.50000	9. 0.78005
10. 2.80002	11. 2.00325	12. 8.99984 − 10

65. Use of Logarithms. The following examples illustrate the use of logarithms in arithmetic computation.

Example 1. Find the value of N, given that

$$N = \frac{(389.2)(48.756)}{8976}.$$

Solution. The work may be arranged as follows:

$$\begin{aligned} \log 389.2 &= 2.59017 \\ \log 48.756 &= \underline{1.68802} \\ \log (389.2)(48.756) &= 4.27819 \qquad \text{(By I, § 59)} \\ \log 8976 &= \underline{3.95308} \\ \log N &= 0.32511 \qquad \text{(By II, § 59)} \\ N &= 2.1140 \end{aligned}$$

In using logarithms, time is saved and errors are avoided by making a so-called *form* for all the numerical work before referring to the table for the values of the logarithms. Thus, in Example 1, the form is

$$\begin{aligned} \log 389.2 &= \\ \log 48.756 &= \underline{\qquad\quad} \\ \log (389.2)(48.756) &= \\ \log 8976 &= \underline{\qquad\quad} \\ \log N &= \\ N &= \end{aligned}$$

Example 2. Find the value of x, if $x = (0.678)^3$.

Solution. The work is arranged as follows:

$$\log (0.678) = 9.83123 - 10$$
$$\log x = 3 \log (0.678) = 29.49369 - 30$$
$$\log x = 9.49369 - 10$$
$$x = 0.31166$$

EXAMPLE 3. Find y, if $y = \sqrt[3]{0.079674}$.

SOLUTION. We recall that $\sqrt[3]{0.079674} = (0.079674)^{1/3}$.

$$\log (0.079674) = 8.90131 - 10$$
$$\log y = \frac{1}{3}(8.90131 - 10).$$

In this case, division by 3 is necessary, but -10 is not exactly divisible by 3. It will facilitate the computation at this point to add and subtract such a multiple of 10 that -10 will appear in the quotient. Thus, we add and subtract 20, giving

$$\log y = \frac{1}{3}(28.90131 - 30)$$
$$\log y = 9.63377 - 10$$
$$y = 0.43030$$

When computing with negative numbers, make the computations with logarithms of the corresponding positive numbers, and finally attach the appropriate sign. Thus, if Example 3 were changed to find $y = \sqrt[3]{-0.079674}$, we should proceed as above, but finally write the result as

$$y = -0.43030$$

EXERCISES

Find the value, to five significant figures, of the unknown in each of the following cases.

1. $x = (76)(8.752)$.
2. $y = (78.96)/(807.5)$.
3. $P = \sqrt{131}$.
4. $Q = \sqrt{0.6784}$
5. $R = \dfrac{(68.762)(78.925)}{(47.56)(9.872)}$.
6. $x = \sqrt{0.2}$
7. $y = \sqrt{0.02}$
8. $z = \sqrt[3]{34}$.
9. $t = \sqrt[3]{-0.77}$
10. $r = (\sqrt{3})(\sqrt[3]{40})(\sqrt[4]{50})$.
11. $x = (\sqrt{0.3})(\sqrt[3]{0.041})(\sqrt[4]{0.0061})$.
12. $y = \dfrac{(\sqrt{875.64})(\sqrt{65.876})}{\sqrt[3]{10027}}$
13. $M = \dfrac{(86.7)\sqrt{7.867}}{95.67}$
14. $N = \dfrac{(6.767)(42.682)}{\sqrt[3]{676.85}}$
15. $r = \dfrac{(78)^{2/3}(98)^{2/5}}{(158)^{3/2}}$
16. $\dfrac{x}{100} = (1.067)^{1.4}$

17. $y = (101)(1.075)^{1.42}$

18. $P = \dfrac{(37.682)(0.46752)}{(0.7654)^{1.4}}$

19. $\sqrt{t} = \dfrac{(0.007867)(0.086752)}{0.56787}$

20. $P^2 = \dfrac{(56.752)(8.6752)}{\sqrt{1456.74}}$

21. $\sqrt[3]{x} = -(17)(0.4367)^{3/2}$

22. $y^3 = (\log 3.7685)^2$

23. $\sqrt[4]{x} = \dfrac{(14.6)\sqrt{0.6789}}{75}$

24. $\dfrac{145x}{75.6} = -8.6752$

25. $r = \dfrac{(1.4)^{1.4}(58.6)}{(16)^{2/3}}$

26. $A = \pi r^2$, $\pi = 3.1416$, $r = 4.87$

27. $V = \frac{4}{3}\pi r^3$, $\pi = 3.1416$, $r = 28.7$

28. The volume of a right circular cone is given by

$$V = \frac{1}{3}\pi r^2 h,$$

where r is the radius of the base and h is the altitude. Find V in cubic feet, if $r = 25$ feet, 4 inches, and $h = 7.5$ yards.

29. Using the formula in Exercise 28, find r given $V = 4000$ cu. in. and $h = 3$ feet.

30. The area S of an equilateral triangle of side a is given by the formula

$$S = \frac{a^2}{4}\sqrt{3}.$$

Find S if $a = 276.75$ feet.

31. Using the formula in Exercise 30, find the length of the side of an equilateral triangle whose area is 1000 square yards.

32. The area S of a triangle whose three sides are given is

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \qquad 2s = a + b + c.$$

Find S, if $a = 175$, $b = 237.5$, $c = 240$.

33. Using Exercise 32, find S if $a = 1375$, $b = 1465$, $c = 2078$.

34. In § 47, Exercise 12, we found the radius r of the inscribed circle of a triangle ABC to be given by the formula

$$r^2 = (s-a)(s-b)(s-c)/s, \qquad 2s = a + b + c.$$

Find r for the triangle whose sides are 76, 91, and 101.

35. Find r for the triangle whose sides are 786.5, 896.5, and 900.6. Use formula of Exercise 34.

36. The radius R of the circle circumscribing a triangle ABC is given by the relation

$$R = \frac{abc}{4S},$$

where a, b, c are the sides and S is the area. Find R for the triangle whose sides are 1324, 2467, and 3256.

HINT. The formula for S is found in Exercise 32.

37. For a pendulum it is known that

$$t = \pi\sqrt{\frac{l}{g}},$$

where t is the time of a complete vibration, l is the length of the pendulum, and g is the acceleration. Find the length l of a pendulum that completes a vibration in a second if $g = 982$ cm. per sec. per sec., and $\pi = 3.1416$.

38. What is the time t of vibration of a pendulum 50 feet long if $g = 32.16$ feet per sec. per sec.?

39. The time t of a vibration of a pendulum is 5/4 seconds. If $g = 986.96$ cm. per sec. per sec., how long is the pendulum?

40. The plate current I in a three-electrode vacuum tube is given by the formula

$$I = 0.00445V^{1.735},$$

where V is the voltage. Find I for $V = 110$.

66. Explanation of the Logarithmic Table III. Logarithms of the Trigonometric Functions. On pages 27–72 we give a five-place table of the logarithms of the trigonometric functions. While Table III gives the logarithms of the functions directly from the angle, these logarithms are, of course, merely the logarithms of the values of the functions. Thus, for example,

$$\log \sin 60° = \log \tfrac{1}{2}\sqrt{3}.$$

Table III gives logarithms of functions at intervals of 1′ from 0° to 90°.

We have given in Table III the value of each logarithm of a trigonometric function increased by 10. Thus, 10 must be subtracted from each tabular value to get the logarithm.

67. The Logarithm of a Trigonometric Function of a Given Angle.

Example 1. Find log sin 36° 29′.

Turn to the page that has 36° at the top. Locate the intersection of the row that has 29′ in the first column with the column that has **L sin** at the top. We thus find that

$$\log \sin 36° 29' = 9.77422 - 10.$$

Example 2. Find log cot 81° 19′.

Turn to the page that has 81° at the bottom. Locate the intersection of the row that has 19′ in the last column with the column that has **L ctn** * at the bottom. We thus find that

$$\log \cot 81° 19' = 9.18391 - 10.$$

* In Table III, ctn is used for cotangent.

EXAMPLE 3. Find log tan 14° 24.6′.

This logarithm is not given directly in the table, but it may be approximated from logarithms in the table by interpolation.

Turn to the page that has 14° at the top. We find on that page that

$$\begin{aligned}\log \tan 14^\circ 24' &= 9.40952 - 10\\ \log \tan 14^\circ 25' &= \underline{9.41005 - 10}\\ \text{Difference} &= 0.00053\\ \log \tan 14^\circ 24.6' &= 9.40952 - 10 + (0.6 \text{ of } 0.00053)\\ &= 9.40952 - 10 + 0.00032\\ &= 9.40984 - 10.\end{aligned}$$

EXAMPLE 4. Find log cos 73° 47.7′.

This may be found approximately as in Example 3. Turn to the page that has 73° at the bottom. We find on that page that

$$\begin{aligned}\log \cos 73^\circ 47' &= 9.44602 - 10\\ \log \cos 73^\circ 48' &= \underline{9.44559 - 10}\\ \text{Difference} &= 0.00043\\ \log \cos 73^\circ 47.7' &= 9.44602 - 10 - (0.7 \text{ of } 0.00043)\\ &= 9.44602 - 10 - 0.00030\\ &= 9.44572 - 10\end{aligned}$$

NOTE. The student should notice in Table III that log sin θ and log tan θ increase as the angle θ increases, but that log cot θ and log cos θ decrease as the angle θ increases.

It follows that, if we always interpolate from the smaller angle, the correction for tenths of a minute will be added in the case of log sin θ and log tan θ, and subtracted in the case of log cot θ and log cos θ.

68. The Angle Corresponding to a Given Logarithm of a Trigonometric Function of an Angle.

EXAMPLE 1. Find the angle θ corresponding to

$$\log \cot \theta = 8.97825 - 10.$$

Turn to the page in the table that has the logarithm 8.97825 in a column having ***L* ctn** at the bottom. (It is not found in a column having ***L* ctn** at the top, as the student should verify.)

This page has 84° at the bottom. The given logarithm is at the intersection of the **L ctn** column, and the row that has 34′ in the last column.

Hence $\theta = 84° \; 34'$ when $\log \cot \theta = 8.97825 - 10$.

EXAMPLE 2. Find the angle θ corresponding to

$$\log \sin \theta = 9.68433 - 10.$$

This logarithm is not given in the table, but the angle may be approximated by interpolation. The logarithm lies between two adjacent logarithms in the column that has **L sin** at the top, on the page that has 28° at the top. We find on that page that

$$\begin{aligned} \log \sin 28° \; 54' &= 9.68420 \\ \log \sin 28° \; 55' &= 9.68443 \\ \text{Difference} &= 0.00023 \end{aligned}$$

We also see that the difference between log sin 28° 54′ and the given logarithm is 0.00013. Hence

$$\theta = 28° \; 54' + \frac{13}{23} \text{ of } 1' = 28° \; 54.6'$$

EXERCISES

Find each of the following five-place logarithms from Table III.

1. log sin 13° 27′
2. log cos 40° 58′
3. log tan 48° 37′
4. log cot 68° 11′
5. log sin 78° 55.3′
6. log cos 65° 8.2′
7. log tan 38° 17.8′
8. log cot 13° 15.3′
9. log sin 58° 13.4′
10. log sin 78° 1′
11. log cos 37° 56.6′
12. log cos 5° 59.5′
13. log tan 75° 11.9′
14. log tan 12° 14.5′
15. log cot 5° 32.4′
16. log cot 44° 44.9′
17. log sin 118° 25.1′ HINT. sin 118° 25.1′ = cos 28° 25.1′
18. log sin 135° 18.3′
19. log tan 227° 28.6′
20. log sin 156° 47.6′
21. log cos 324° 45.3′
22. log tan 256° 30.5′
23. log cot 200° 15.3′

Find the acute angle A, in each case, from the information given below.

24. $\log \sin A = 9.68716 - 10.$
25. $\log \cos A = 9.78658 - 10.$
26. $\log \tan A = 9.75382 - 10.$
27. $\log \cot A = 0.32281$
28. $\log \sin A = 9.84025 - 10.$
29. $\log \cos A = 9.88960 - 10.$
30. $\log \tan A = 0.74540$
31. $\log \cot A = 9.21800 - 10.$
32. $\log \sin A = 9.87654 - 10.$
33. $\log \sin A = 9.45123 - 10.$

34. $\log \cos A = 9.12345 - 10.$

35. $\log \cos A = 9.00000 - 10.$

36. $\log \tan A = 0.23467$

37. $\log \tan A = 9.78650 - 10.$

38. $\log \cot A = 9.00000 - 10.$

39. $\log \cot A = 0.00000$

Find two positive angles A less than 360°, in each case, from the information given below.

40. $\log \sin A = 9.53010 - 10.$

41. $\log \cos A = 9.98038 - 10.$

42. $\log \tan A = 0.31352$

43. $\log \cot A = 9.65124 - 10.$

44. $\log \sin A = 9.80007 - 10.$

45. $\log \cos A = 9.18888 - 10.$

46. $\log \tan A = 9.64289 - 10.$

47. $\log \cot A = 9.24680 - 10.$

48. The area of a triangle is given by

$$S = \tfrac{1}{2}ab \sin C.$$

Find S if $a = 76.2$, $b = 36.5$, $C = 63° 18'$.

49. Find S if $b = 85$, $c = 75$, $A = 100° 21.5'$. Use the formula in Exercise 48.

50. The area S of a triangle is given by

$$S = c^2 \sin A \sin B/(2 \sin C).$$

Find S, given $c = 778$, $A = 68° 17'$, $B = 31° 56'$.

51. Find S for the triangle in which $c = 101.34$, $B = 60° 13.4'$, and $C = 75° 23.5'$.

Hint. See Exercise 50.

52. The volume of a right circular cone is given by

$$V = \tfrac{1}{3}\pi h^3 \tan^2 A,$$

where h is the altitude and A is the generating angle. Find V, if $h = 8.5$, $A = 37° 5.3'$, and $\pi = 3.1416$.

53. From the law of sines, $a = b \sin A/\sin B$. Find a, given $b = 375.21$, $A = 36° 7.2'$, and $B = 65° 55.7'$.

69. Accuracy of a Five-Place Table. Five-place tables are sufficiently accurate to satisfy the requirements of the ordinary data of engineering and physical science. A measurement expressed by a five-place number presupposes a relative accuracy of at least 1/200 of 1%.

An angle may very commonly be determined from a five-place table of logarithms of its functions with an error of not more than one-tenth minute, but the accuracy of this determination depends much on the size of the angle and on the function from which it is determined.

In the answers to exercises and problems in this book, which are found by calculations with five-place tables, we adopt the practice of ordinarily giving answers showing angles to tenths of a minute and distances to five significant figures although this degree of accuracy is not always warranted.

70. Reporting Angles in Degrees, Minutes, and Seconds. While we usually report answers in degrees, minutes, and tenths of a minute, we shall give some exercises for drill on the practice of reporting angles to degrees, minutes, and seconds.

EXERCISES

Express each of the following angles in degrees, minutes, and seconds.

1. $19° 21.4'$ **2.** $87° 39.2'$ **3.** $50° 43.7'$
4. $27° 10.3'$ **5.** $41° 41.8'$ **6.** $65° 29.6'$

Express each of the following angles in degrees, minutes, and tenths of a minute.

7. $17° 19' 25''$ **8.** $81° 34' 55''$ **9.** $46° 41' 17''$
10. $20° 7' 31''$ **11.** $39° 40' 43''$ **12.** $62° 26' 4''$

Find each of the following five-place logarithms from Table III.

13. $\log \sin 17° 32' 13''$

SOLUTION.

(a) $\log \sin 17° 32' = 9.47894 - 10,$
(b) $\log \sin 17° 32' 13'' = x,$
(c) $\log \sin 17° 33' = 9.47934 - 10.$

We get (a) and (c) from Table III, p. 45, and are to find x in (b) by interpolation. The difference between the two mantissas is 40. Since $32' 13''$ is 13/60 of the way from $32'$ to $33'$, we add $(13/60)(40) = 9$ to the mantissa 47894. This gives

$$\log \sin 17° 32' 13'' = 9.47903 - 10.$$

14. $\log \sin 23° 19' 18''$ **15.** $\log \cos 48° 21' 26''$
16. $\log \tan 27° 39' 50''$ **17.** $\log \cot 21° 33' 21''$
18. $\log \tan 52° 44' 31''$ **19.** $\log \cos 81° 39' 36''$
20. $\log \sin 51° 9' 47''$ **21.** $\log \sin 80° 12' 21''$

Find the acute angle A by the use of the five-place Table III, reporting the results in degrees, minutes, and seconds.

22. $\log \sin A = 9.77118 - 10.$

SOLUTION.

(a) $\log \sin 36° 11' = 9.77112 - 10,$
(b) $\log \sin A = 9.77118 - 10,$
(c) $\log \sin 36° 12' = 9.77130 - 10.$

We get (a) and (c) from Table III, p. 64, and are to find A in (b). By proportional parts, we see that A is 1/3 of the way from $36° 11'$ to $36° 12'$. Hence

$$A = 36° 11' + \frac{1}{3} \text{ of } 1' = 36° 11' 20''.$$

23. $\log \sin A = 9.77821 - 10.$ **24.** $\log \cos A = 8.99365 - 10.$
25. $\log \tan A = 0.20722$ **26.** $\log \cot A = 0.10262$
27. $\log \cos A = 9.78264 - 10.$ **28.** $\log \sin A = 8.99264 - 10.$
29. $\log \tan A = 9.48560 - 10.$ **30.** $\log \cot A = 9.72159 - 10.$

71. Angles Near 0° and 90°. Table III is inadequate for obtaining by interpolation the values of log sin θ, log tan θ, and log cot θ when θ is near 0°, and likewise the values of log cos θ, log cot θ, and log tan θ when θ is near 90°. To obtain such values the student may use a table with angles tabulated at smaller intervals.

Table III is also inadequate to obtain the angle θ from log cos θ when θ is near 0°, and from log sin θ when θ is near 90°.

72. Natural Logarithms. In our study of logarithms, we have considered only logarithms to the base **10**. For a more general definition we may say that, if

(1) $$N = a^x,$$

where N, a, and x are any three numbers ($a > 0$, $a \neq 1$), then the exponent x is the *logarithm of N to the base a, and we write* as the equivalent of (1)

(2) $$x = \log_a N.$$

It is fairly obvious that the properties of logarithms found in § 59 can be derived for any *base* a by substituting a for 10.

EXAMPLE. Prove $\log_a mn = \log_a m + \log_a n$.

For purposes of computation, 10 is the most convenient base. Why?

For purposes of the calculus and in higher mathematics a base called e where

$$e = 2.71828 \cdots,$$

is more convenient than base 10. Logarithms to the base e are called ***natural logarithms*** to distinguish them from common logarithms.

CHAPTER IX

SOLUTION OF TRIANGLES BY LOGARITHMS

73. Introduction. In this chapter, logarithms will be used in the solution of triangles. We shall find it convenient to treat right and oblique triangles separately.

74. Right Triangles. The formulas for solving a right triangle, ABC (Fig. 86), are as follows:

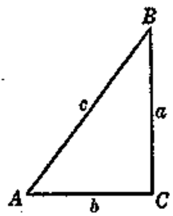


Fig. 86

(a) $$\sin A = \frac{a}{c} = \cos B,$$

(b) $$\cos A = \frac{b}{c} = \sin B,$$

(c) $$\tan A = \frac{a}{b} = \cot B,$$

(d) $$A + B = 90^\circ,$$

(e) $$a^2 + b^2 = c^2.$$

The formulas (a), (b), and (c) are well adapted to logarithmic computation. Consider the following illustrative examples.

Example 1. Given $c = 176.4$, $A = 18^\circ\ 21.5'$, find a, b, and B.

Solution. The values of the known parts are indicated in Fig. 87.

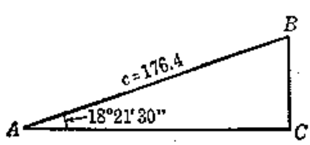


Fig. 87

Formulas:

$$a = c \sin A, \qquad b = c \cos A, \qquad B = 90^\circ - A = 71^\circ\ 38.5'$$

$$\begin{aligned} \log c &= 2.24650 \\ \log \sin A &= 9.49825 - 10 \\ \log a &= 11.74475 - 10 \\ a &= 55.559 \end{aligned} \qquad \begin{aligned} \log c &= 2.24650 \\ \log \cos A &= 9.97732 - 10 \\ \log b &= 12.22382 - 10 \\ b &= 167.42 \end{aligned}$$

EXAMPLE 2. Given $c = 1145$, $a = 1064$, find b, A, and B.

SOLUTION. The values of the known parts are indicated in Fig. 88.

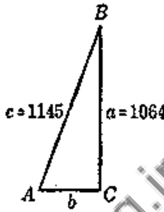


FIG. 88

FORMULAS:

$$\sin A = \frac{a}{c}, \quad b = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)},$$

$$B = 90° - A.$$

$$\begin{aligned} \log a &= 13.02694 - 10 \\ \log c &= 3.05881 \\ \log \sin A &= 9.96813 - 10 \\ A &= 68° \, 19', \\ B &= 21° \, 41'. \end{aligned}$$

$$\begin{aligned} c &= 1145 \\ a &= 1064 \\ c + a &= 2209 \\ c - a &= 81 \\ \log (c + a) &= 3.34420 \\ \log (c - a) &= 1.90849 \\ \log b^2 &= 5.25269 \\ \log b &= 2.62634 \\ b &= 423.00 \end{aligned}$$

We use $b = \sqrt{c^2 - a^2}$ in preference to $b = c \cos A$ in accord with the practice of using the original data rather than computed parts in finding other parts.

EXERCISES

Solve the following right triangles.

1. $c = 2279$, $A = 46° \, 27'$
2. $a = 675$, $b = 423$.
3. $a = 130$, $c = 200$.
4. $a = 520$, $A = 43° \, 27.5'$
5. $b = 292.4$, $c = 982.7$
6. $c = 1109.7$, $a = 672.91$
7. $b = 1843.6$, $B = 65° \, 15.6'$

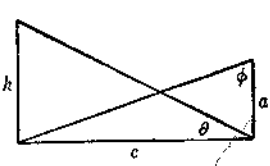


FIG. 89

8. In Fig. 89, show that $h = a \tan \phi \tan \theta$.
9. From the bottom of a building 36 feet high, the angle of elevation of the top of a radio tower is 65° 35.7′. From the top of the building the angle of depression of the base of the radio tower is 48° 31.5′. Find the height of the radio tower. Use the formula in Exercise 8.
10. At a point A, the angle of elevation of the top of a monument due west is 60° 15.5′. At a point B, 352 feet due south of A, the base of the monument bears N 37° 16.8′ W. Find the height of the monument above the horizontal plane containing the points A and B.

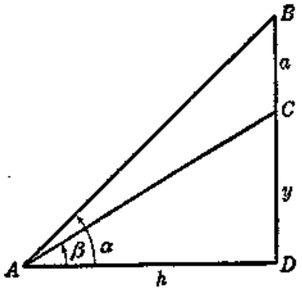


FIG. 90

11. In Fig. 90, show that

$$a = \frac{h \sin (\alpha - \beta)}{\cos \alpha \cos \beta}.$$

HINT. Note that $a = h \tan \alpha - h \tan \beta$.

12. If D falls between B and C in Exercise 11, show that

$$a = \frac{h \sin (\alpha + \beta)}{\cos \alpha \cos \beta}$$

$$= \frac{h \sin A}{\sin B \sin C}.$$

13. Find the height of a flagpole standing on a building if at a horizontal distance of 184.2 feet from the flagpole, the angles of elevation of its top and bottom are 59° 46′ and 49° 33′ respectively. Use the formula of Exercise 11.

14. Find the altitude from vertex A of the triangle ABC (Fig. 90), given $a = 18.75$, $B = 35° 17.3'$, and $C = 105° 32.8'$. Use the formula of Exercise 11.

15. At a horizontal distance of 1000 feet from a smoke stack, the angle of elevation of the top of the stack is 9° 20.5′ and the angle of depression of the bottom 10°, how high is the smoke stack? Use the formula of Exercise 12.

75. Oblique Triangles. In Chapter VI, the law of sines and the law of cosines were used to compute the unknown parts of a triangle. In general, if three of the six parts of a triangle are given, provided the given parts are not all angles, the remaining parts can be found. The process of finding these unknown parts is called the *solution* of the triangle.

Four cases arise, according as the given parts are:

CASE I. *One side and two angles.*
CASE II. *Two sides and the angle opposite one of them.*
CASE III. *Two sides and their included angle.*
CASE IV. *Three sides.*

It is assumed that the data are consistent with the two following conditions.

(1) $$A + B + C = 180°.$$

(2) *Any side of a triangle is less than the sum of the other two sides.*

In solving triangles which belong to Cases I and II, we need no new theorems. We simply use the law of sines as in § 43 but make our multiplications and divisions by the use of logarithms.

76. Case I. *Given one side and two angles.*

The following example will illustrate how logarithms may be used in the solution.

EXAMPLE 1. Given that $c = 18.26$, $A = 55° 18'$, $B = 65° 18.6'$, find C, a, and b.

SOLUTION. The values of the known parts are indicated in Fig. 91.

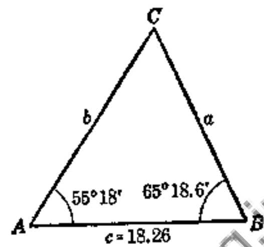


FIG. 91

FORMULAS:

$$C = 180° - A - B = 59° 23.4',$$

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\log c = 11.26150 - 10$$
$$\log \sin C = 9.93483 - 10$$
$$\log\left(\frac{c}{\sin C}\right) = 1.32667$$
$$\log \sin A = 9.91495 - 10$$
$$\log \sin B = 9.95837 - 10$$
$$\log a = 11.24162 - 10$$
$$\log b = 11.28504 - 10$$
$$a = 17.443$$
$$b = 19.277$$

In solving triangles, time is saved and errors avoided by making a so-called form for all numerical work before using tables on any part of the work. Compare remark and illustration in § 65.

EXERCISES

Solve each of the following triangles.

1. $b = 1926$,	$A = 5° 26.2'$,	$B = 72° 36.8'$.
2. $a = 143.2$,	$A = 53° 17.3'$,	$B = 62° 23.5'$.
3. $a = 4126.7$,	$A = 50° 38.9'$,	$B = 60° 7.4'$.
4. $c = 3843$,	$B = 75° 28'$,	$C = 66° 45'$.
5. $b = 728.36$,	$B = 54° 21.3'$,	$C = 68° 15.5'$.
6. $c = 250.4$,	$B = 15° 19.8'$,	$C = 72° 44.1'$.

7. From two points 481 yards apart on a straight level road, running due east, the bearings of an oil derrick are found to be N 47° 55.3′ E and N 71° 8.6′ W. Find the distance of the derrick from each point.

8. To find the distance from a point A on one side of a lake to a point C on the other side, the following observations were made. A distance $AB = 2377$ feet was taken and the angles BAC and ABC were found to be 37° 24′ and 65° 47.5′ respectively. Find the length of AC.

77. Case II. *Given two sides and the angle opposite one of them.* The data for triangles under Case II may be such that:

(a) One triangle fits the data.
(b) Two triangles fit the data.
(c) No triangle fits the data.

Assume that the given parts are a, b, and A.

Illustrative drawings. Figures 92–97 with the restrictions on A, a, and b stated below each figure, illustrate conditions under which there is one solution, two solutions, and no solution.

Figures 92–94 indicate conditions under which there is one solution.

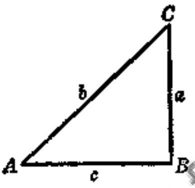


Fig. 92
A acute
$a = b \sin A$

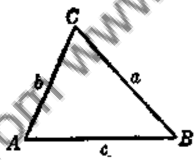


Fig. 93
A acute
$a \geqq b$

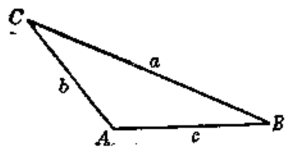


Fig. 94
A obtuse
$a > b$

Figure 95 indicates conditions under which there are two solutions. Figures 96 and 97 indicate conditions under which there are no solutions.

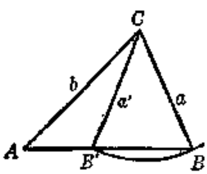


Fig. 95
A acute
$a < b, \quad a > b \sin A$

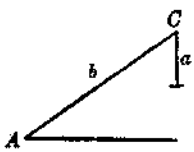


Fig. 96
A acute
$a < b \sin A$

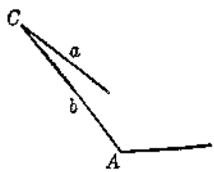


Fig. 97
A obtuse
$a < b$

Case II is commonly referred to as the ***ambiguous case.*** A carefully constructed figure will usually indicate the number of solutions.

EXAMPLE 1. Given $a = 150$, $c = 170$, $A = 61° 55.6'$, find b, B, and C.

SOLUTION. Construct Fig. 98 indicating the values of known parts.

FORMULA:

$$\frac{\sin C}{c} = \frac{\sin A}{a}.$$

$$\begin{aligned} \log c &= 2.23045 \\ \log \sin A &= 9.94564 - 10 \\ \log c \sin A &= 12.17609 - 10 \\ \log a &= 2.17609 \\ \log \sin C &= 0.00000 \\ C &= 90°. \end{aligned}$$

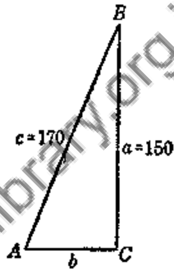


FIG. 98

Since C is a right angle, there is only one solution; we have then

$$B = 28° 4.4'$$

Completing the solution, we have

$$\begin{aligned} b &= c \cos A \\ \log c &= 2.23045 \\ \log \cos A &= 9.67266 - 10 \\ \log b &= 11.90311 - 10 \\ b &= 80.004 \end{aligned}$$

EXAMPLE 2. Given $b = 3400$, $c = 2200$, $C = 30° 20'$, find a, A, and B.

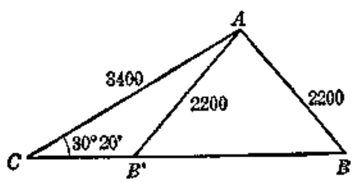


FIG. 99

SOLUTION. Draw Fig. 99 to scale indicating the values of known parts. The figure indicates that there are two solutions, triangles ABC and $AB'C$. This can be verified by noting that angle C is acute and $c < b$, and then proving that $c > b \sin C$. These are the conditions stated under Fig. 95.

To prove $c > b \sin C$ and to find B, we use the formula

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

$$\begin{aligned} \log b &= 3.53148 \\ \log \sin C &= 9.70332 - 10 \\ \text{(a)} \quad \log b \sin C &= 13.23480 - 10 \\ \text{(b)} \quad \log c &= 3.34242 \\ \log \sin B &= 9.89238 - 10 \\ B &= 51^\circ\ 18.5' \end{aligned}$$

This gives angle B of the triangle ABC. From (a) and (b) we note also that $c > b \sin C$.

We now complete the solution of triangle ABC.

FORMULAS: $A = 180° - B - C = 98°\ 21.5'$,

$$a = \frac{b \sin A}{\sin B}.$$

$$\begin{aligned} \log b &= 3.53148 \\ \log \sin A &= 9.99536 - 10 \\ \log b \sin A &= 13.52684 - 10 \\ \log \sin B &= 9.89238 - 10 \\ \log a &= 3.63446 \\ a &= 4309.8 \end{aligned}$$

We now complete the solution of the triangle $AB'C$. The angle B' in the triangle $AB'C$ is $B' = 180° - B = 128°\ 41.5'$.

FORMULAS: $A = 180° - B' - C = 20°\ 58.5'$,

$$a = \frac{b \sin A}{\sin B'}.$$

$$\begin{aligned} \log b &= 3.53148 \\ \log \sin A &= 9.55383 - 10 \\ \log b \sin A &= 13.08531 - 10 \\ \log \sin B' &= 9.89238 - 10 \\ \log a &= 3.19293 \\ a &= 1559.3 \end{aligned}$$

EXAMPLE 3. Given $c = 690$, $A = 47°\ 16'$, $a = 450$, find b, B, and C.

SOLUTION. From Fig. 100 drawn to scale, it appears that no solution is possible.

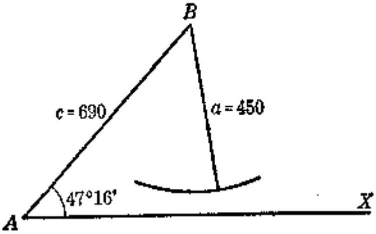


FIG. 100

To prove that no solution is possible, we proceed as follows.

FORMULA:

$$\sin C = \frac{c \sin A}{a}.$$

$$\begin{aligned} \log c &= 2.83885 \\ \log \sin A &= 9.86600 - 10 \\ \log c \sin A &= 12.70485 - 10 \\ \log a &= 2.65321 \\ \log \sin C &= 10.05164 - 10. \end{aligned}$$

Since $\log \sin C$ is greater than zero, it follows that $\sin C$ is greater than one. But $\sin C$ is never greater than one; hence the angle C cannot be found. Therefore, there is no solution.

It is left as an exercise for the student to find the length of the perpendicular from B to AX which equals $c \sin A$. Note that this length is greater than 450.

EXERCISES

Solve each of the following triangles.

1. $b = 350$, $c = 400$, $B = 40°$
2. $a = 274.95$, $b = 340$, $A = 53° 58'$
3. $a = 180$, $c = 100$, $A = 127° 33'$
4. $b = 600$, $c = 250$, $C = 42° 12'$
5. $a = 512.3$, $b = 423.6$, $A = 55°$
6. $a = 40$, $b = 50$, $A = 53° 7.8'$
7. $a = 700$, $c = 400$, $C = 47° 32.2'$
8. $a = 235$, $b = 189$, $B = 36° 28.3'$
9. $a = 434.77$, $b = 400.31$, $A = 94° 17.6'$
10. $a = 946.72$, $c = 1443.3$, $A = 11° 14.3'$
11. $b = 10.23$, $c = 3.921$, $B = 16° 55'$

12. In running a course, a surveyor encounters a barrier. He alters his course by an angle 31° 10.5′ and goes a distance of 678.4 yards. He then finds that he can most easily get back to his original course by going 556 yards. Find the distance or distances in the original course not traversed by the surveyor.

13. Two observers, 4000 feet apart, observe simultaneously the angles of elevation of a balloon. If the balloon is in the vertical plane of the observers, and if the angles of elevation are 47° 38′ and 58° 18′, find the height of the balloon. Two solutions.

78. Law of Tangents. Before taking up the solution of triangles under Case III, it is desirable to develop the law of tangents.

In any triangle ABC $(a > b)$,

$$\frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} = \frac{a-b}{a+b}.$$

This relation was given in Exercise 14, § 44. We shall now give another proof.

Proof. From the law of sines, we may write

$$\frac{\sin A}{\sin C} = \frac{a}{c}, \qquad \frac{\sin B}{\sin C} = \frac{b}{c}.$$

By subtracting and adding, we have

(a) $$\frac{\sin A - \sin B}{\sin C} = \frac{a-b}{c},$$

(b) $$\frac{\sin A + \sin B}{\sin C} = \frac{a+b}{c}.$$

Dividing (a) by (b) member by member, we find

(c) $$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b}.$$

By (21) and (22), § 55,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}$$

$$= \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}},$$

whence

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{a-b}{a+b}.$$

Similarly, we may derive the formulas

$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\frac{b-c}{b+c},\qquad \frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}}=\frac{c-a}{c+a}.$$

Stated in words, the law of tangents is: *In any triangle, the tangent of half the difference of any two angles is to the tangent of half their sum as the difference of the sides opposite these angles is to their sum.*

79. Case III. *Given two sides and the included angle.*

The law of tangents is well adapted to the solution of triangles by logarithms when two sides and the included angle are given, whereas the law of cosines used in § 45 is not well adapted to logarithmic computation. The following example will serve as an illustration:

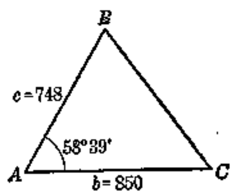


FIG. 101

EXAMPLE. Given $b = 850$, $c = 748$, and $A = 58° 39'$, find a, B, and C.

SOLUTION. The values of known parts are indicated in Fig. 101.

FORMULAS: $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2},$$

$$a=\frac{b\sin A}{\sin B}.$$

$$b-c = 102$$
$$b+c = 1598$$
$$B + C = 180 - A = 121° 21'$$

(a) $$\frac{B+C}{2} = 60° 40.5'$$

$$\log(b-c) = 2.00860$$
$$\log\tan\frac{B+C}{2} = 0.25046$$

$$\log (b - c) \tan \frac{B+C}{2} = 12.25906 - 10$$
$$\log (b + c) = 3.20358$$
$$\log \tan \frac{B-C}{2} = 9.05548 - 10$$

Then $\dfrac{B-C}{2} = 6° \ 29.0'$

But from (a) $\dfrac{B+C}{2} = 60° \ 40.5'$

Add, $B = 67° \ 9.5'$

Subtract, $C = 54° \ 11.5'$

$$\log b = 2.92942$$
$$\log \sin A = 9.93146 - 10$$
$$\log b \sin A = 12.86088 - 10$$
$$\log \sin B = 9.96453 - 10$$
$$\log a = 2.89635$$
$$a = 787.68$$

EXERCISES

Solve each of the following triangles.

1. $a = 304$, $b = 280.3$, $C = 100°$.
2. $b = 300$, $c = 95.6$, $A = 65° \ 47.9'$
3. $a = 500$, $c = 395.75$, $B = 52° \ 0.5'$
4. $a = 312$, $b = 423$, $C = 52° \ 30.5'$
5. $b = 813.96$, $c = 993.67$, $A = 67° \ 27.1'$
6. $a = 2236$, $c = 2646$, $B = 59° \ 30.8'$

7. Two forces of 450 and 540 dynes act simultaneously at a point. If the angle between the forces is 80°, find the magnitude of their resultant.

8. Two airplanes leave an airport at the same time. The first plane flies a course S 83° 5.4′ E at a rate of 150 miles per hour. The second plane flies a course N 55° 30′ E at a rate of 125 miles per hour. How far apart are the airplanes in one hour?

80. Tangents of Half-Angles. Before taking up the solution of triangles belonging to Case IV, it is desirable to develop a formula for the tangent of one-half of an angle of the triangle.

In any triangle, ABC, we have

(1) $$\tan \frac{A}{2} = \frac{r}{s-a},$$

where $2s = a + b + c$ and $r = \sqrt{(s-a)(s-b)(s-c)/s}$.

PROOF. In the triangle ABC (Fig. 102), let O be the center of the inscribed circle. It then follows that AO, BO, and CO bisect the angles A, B, and C respectively. Also we know that

$$AN = MA, \qquad BL = NB, \qquad LC = CM,$$

and, by addition, we find

(a) $$AN + BL + LC = MA + NB + CM.$$

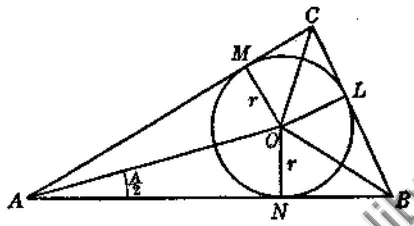


FIG. 102

Since the perimeter of the triangle is $2s$, each side of (a) is equal to s, that is,

$$AN + BL + LC = s, \quad \text{and} \quad AN + a = s.$$

Hence

$$AN = s - a.$$

From the right triangle ANO, we have

$$\tan\frac{A}{2} = \frac{r}{s-a}.$$

In a similar manner, we may show that

(2) $$\tan\frac{B}{2} = \frac{r}{s-b},$$

(3) $$\tan\frac{C}{2} = \frac{r}{s-c}.$$

It remains to show that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

From Fig. 102,

$$\text{Area } ABC = \text{area } ABO + \text{area } BOC + \text{area } COA.$$

Then it follows, using S for area ABC, that

(b) $$S = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = rs.$$

Since, by § 47,

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

we find, from (b),

$$rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Hence

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

81. Case IV. *Given three sides.*

In Chapter VI, the law of cosines was used to compute the angles of a triangle when three sides were given. The formulas for the tangents of the half-angles are well adapted to computation by logarithms as will become obvious from the following example.

EXAMPLE. Given $a = 131.6$, $b = 150$, $c = 201$, find A, B, and C.

SOLUTION. The given parts are indicated in Fig. 103.

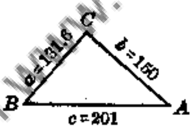


FIG. 103

FORMULAS:

$$\tan\frac{A}{2} = \frac{r}{s-a}, \qquad \tan\frac{B}{2} = \frac{r}{s-b}, \qquad \tan\frac{C}{2} = \frac{r}{s-c},$$

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}, \qquad 2s = a + b + c,$$

$a = 131.6$	$\log(s-a) = 2.04021$
$b = 150$	$\log(s-b) = 1.96047$
$c = 201$	$\log(s-c) = 1.60531$
$2s = 482.6$	sum $= 5.60599$
$s = 241.3$	$\log s = 2.38256$
$s-a = 109.7$	$\log r^2 = 3.22343$
$s-b = 91.3$	$\log r = 1.61172$
$s-c = 40.3$	

We now use the formulas:

$$\tan\frac{A}{2} = \frac{r}{s-a} \qquad\qquad \tan\frac{B}{2} = \frac{r}{s-b}$$

$$\log r = 1.61172 \qquad\qquad \log r = 1.61172$$

$$\log(s-a) = 2.04021 \qquad\qquad \log(s-b) = 1.96047$$

$$\log\tan\frac{A}{2} = 9.57151 - 10 \qquad\qquad \log\tan\frac{B}{2} = 9.65125 - 10$$

$$\frac{A}{2} = 20^\circ\ 26.8' \qquad\qquad \frac{B}{2} = 24^\circ\ 7.9'$$

$$A = 40^\circ\ 53.6' \qquad\qquad B = 48^\circ\ 15.8'$$

$$\tan\frac{C}{2} = \frac{r}{(s-c)}$$

$$\log r = 1.61172$$

$$\log(s-c) = 1.60531$$

$$\log\tan\frac{C}{2} = 10.00641 - 10$$

$$\frac{C}{2} = 45^\circ\ 25.4'$$

$$C = 90^\circ\ 50.8'$$

The computation may be checked by taking the sum of the computed angles. In this case the sum is 180° 0.2′. The failure of the sum of the computed angles to check with 180° by 0.2′ is due to the approximate character of the tables. See § 69. However this check is regarded as satisfactory.

EXERCISES

Solve each of the following triangles.

1. $a = 60$, $b = 50$, $c = 40$.
2. $a = 312$, $b = 423$, $c = 342$.
3. $a = 4238$, $b = 3500$, $c = 2900$.
4. $a = 400$, $b = 340$, $c = 250$.
5. $a = 100$, $b = 120$, $c = 140$.
6. $a = 6053$, $b = 4082$, $c = 7068$.

7. The distances between three oil wells A, B, and C are as follows: $AB = 633$ yards, $AC = 850$ yards, $BC = 785$ yards. If A is due south of C, in what direction is B from C, if B lies east of A and C?

8. An observer is 444 yards from the engine of a freight train and 312 yards from the caboose. If the train is 272 yards long, what angle is subtended by the train at the eye of the observer?

9. Two forces of 1026 and 1824 dynes have a resultant of 1598 dynes. Find the angle between the two forces.

10. The sides of a triangle are 450, 309, and 624.5. Find the altitude upon the longest side.

82. Areas. From § 47, we recall that the area S of a triangle is given by any one of the following formulas:

(a) $$S = \frac{1}{2} ab \sin C.$$

(b) $$S = \frac{1}{2} \frac{a^2 \sin B \sin C}{\sin A}.$$

(c) $$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

While other formulas can be given for the area of a triangle, each of the formulas (a), (b), and (c) is well adapted to logarithmic computation.

EXERCISES

Find the area of each of the following triangles, given:

1. $a = 75.3$,	$A = 90° 50'$,	$B = 73° 10'$.
2. $b = 372$,	$B = 51°$,	$C = 74°$.
3. $a = 25$,	$b = 21.25$,	$C = 45°$.
4. $b = 32.939$,	$c = 52.925$,	$A = 21° 36.2'$
5. $a = 322.7$,	$b = 416.5$,	$c = 552.8$
6. $a = 20$,	$b = 30$,	$c = 40$.

7. Two sides of a triangle are 1000 feet and 2000 feet. If these sides include an acute angle, and if the area is 672930 square feet, find the third side.

HINT. Find the angle between the sides and use the law of cosines.

8. A field $ABCD$ has $AB = 25.63$ rods, $BC = 24.09$ rods, $CD = 9.92$ rods, and $DA = 27.97$ rods. The angle between AB and BC is $78° 25'$. Find the area of the field in acres.

HINT. One acre is 160 square rods.

83. Checking. It is important to check the accuracy of any computation. To check the solution of a triangle, any formula which uses as many as possible of the computed parts is likely to serve as a good check, but such a formula will not necessarily be satisfied to the last figure. We may prove the formula

$$\frac{a+b}{c}=\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$$

for this purpose.

PROOF. From the law of sines, we have

$$\frac{a}{c}=\frac{\sin A}{\sin C}, \qquad \frac{b}{c}=\frac{\sin B}{\sin C}.$$

Adding, we have

$$\text{(a)} \qquad \frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin C},$$

by (21), § 55. But

$$A+B=180^\circ-C, \qquad \frac{A+B}{2}=90^\circ-\frac{C}{2}.$$

Hence

$$\text{(b)} \qquad \sin\frac{A+B}{2}=\sin\left(90^\circ-\frac{C}{2}\right)=\cos\frac{C}{2}$$

and

$$\text{(c)} \qquad \sin C=2\sin\frac{C}{2}\cos\frac{C}{2}.$$

Hence we have from (a), (b), and (c),

$$\frac{a+b}{c}=\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}.$$

By symmetry two other formulas may be written down. We shall refer to all these formulas as *Newton's formulas.*

Newton's formulas serve as a good check since they have not been used in the solution of triangles and since they contain all six parts of the triangle. They are adapted to logarithmic computation.

The following summary gives not only the formulas for the logarithmic solution of triangles but those for checking and for the area as well.

Case	I	II	III	IV
Given:	One side and two angles.	Two sides and an angle opposite one of them.	Two sides and their included angle.	Three sides.
Solution	Law of sines.	Law of sines.	Law of tangents.	Tangents of half-angles.
Check	Newton's formula, or law of tangents.	Newton's formula, or law of tangents.	Newton's formula, or law of sines.	$A + B + C = 180°$
Area	$\frac{a^2 \sin B \sin C}{2 \sin A}$	$\frac{ab \sin C}{2}$	$\frac{ab \sin C}{2}$	$\sqrt{s(s-a)(s-b)(s-c)}$

EXAMPLE. Use Newton's formula to check the computed parts of the triangle in Example 1, § 76.

The given and computed parts are as follows:

Given	Computed
$c = 18.26,$	$C = 59° 23.4',$
$A = 55° 18',$	$a = 17.443,$
$B = 65° 18.6'$	$b = 19.277$

From these we derive $a + b = 36.720$, and

$$\frac{C}{2} = 29° 41.7', \qquad B - A = 10° 0.6', \qquad \frac{B-A}{2} = 5° 0.3'$$

Check formula: $$\frac{b+a}{c} = \frac{\cos\frac{B-A}{2}}{\sin\frac{C}{2}}.$$

$$\log (b + a) = 1.56490, \qquad \log \cos\frac{B-A}{2} = 9.99834 - 10,$$

$$\log c = 1.26150, \qquad \log \sin\frac{C}{2} = 9.69494 - 10,$$

$$\log\frac{b+a}{c} = 0.30340, \qquad \log\frac{\cos\frac{B-A}{2}}{\sin\frac{C}{2}} = 0.30340$$

The final values agree. However, exact agreement to five significant figures is more than we should reasonably expect in computing with five-place tables.

EXERCISES

Use one of Newton's formulas to check the computed parts of the following triangle.

1. Given: $C = 123° 4.6'$, $b = 234.25$, $c = 417.92$
   Computed: $A = 28° 54.6'$, $B = 28° 0.8'$, $a = 241.11$

Use the law of tangents to check the computed parts of the following triangle.

2. Given: $b = 3400$, $c = 2200$, $C = 30° 20'$.
   Computed: $A = 98° 21.6'$, $B = 51° 18.4'$, $a = 4309.8$

Use the law of sines to check the computed parts of the following triangle.

3. Given: $b = 41.02$, $c = 45.49$, $A = 62° 9.6'$
   Computed: $B = 54° 1.2'$, $C = 63° 49.2'$, $a = 44.824$

Solve each of the following triangles, and check the computed parts, given:

4. $A = 47° 19'$, $C = 113° 18'$, $c = 353.55$
5. $a = 623$, $b = 507$, $c = 428$.
6. $A = 77°$, $b = 630.77$, $c = 829.48$

7. Two highways cross at an angle of 47° 30′. Find the distances between two towns, one on each highway, that are 18 and 27 miles from the crossing. Two solutions.

8. Two ships start from a port P, at the same time. One is steaming 20 miles per hour S 12° W and the other is steaming 25 miles per hour N 63° W. How far are they apart in two hours?

9. Two sides of a triangle are 505 feet and 465 feet and the difference of the angles opposite these sides is 16° 20.5′. Solve the triangle.

HINT. Use law of tangents.

10. The diagonals of a parallelogram are 90 feet and 145 feet, and one side is 70 feet. Find the angles of the parallelogram and its remaining side.

MISCELLANEOUS EXERCISES

Solve each of the following right triangles, given the parts stated below.

1. $B = 76° 37'$, $b = 244.91$
2. $A = 62° 27.8'$, $b = 1010.5$
3. $a = 67.534$, $c = 79.728$
4. $a = 20.365$, $b = 15.583$
5. $B = 49° 47.4'$, $c = 950.04$
6. $A = 50° 20.9'$, $b = 235.64$

Solve each of the following triangles, given the parts stated below.

7. $B = 61° 58'$,	$C = 80° 49'$,	$c = 5.6588$
8. $A = 21° 21.5'$,	$C = 58° 17.3'$,	$b = 85.502$
9. $A = 105° 6.7'$,	$B = 60° 14.4'$,	$a = 283.82$
10. $A = 42° 7.7'$,	$C = 107° 15.5'$,	$a = 189.64$
11. $b = 235$,	$c = 196$,	$C = 38° 45.4'$
12. $A = 55°$,	$a = 512.34$,	$b = 423.56$
13. $A = 42° 44.4'$,	$a = 29.000$,	$c = 46.136$
14. $a = 100$,	$b = 120$,	$C = 78° 27.8'$
15. $b = 40.82$,	$c = 70.86$,	$A = 58° 41.8'$
16. $a = 300$,	$c = 500$,	$B = 53° 7.8'$
17. $a = 6000$,	$b = 5000$,	$C = 41° 24.6'$
18. $b = 12915$,	$c = 11206$,	$A = 56° 25.8'$
19. $a = 1686$,	$b = 960$,	$c = 2400$.
20. $a = 760$,	$b = 1090$,	$c = 1448$.
21. $a = 3400$,	$b = 3579$,	$c = 6656$.
22. $a = 16000$,	$b = 26690$,	$c = 24737$.

23. Find the third side of a triangle if two sides which include an acute angle are 50 and 60, and if the area is 1469.7.

24. Given $a = 250$, $b = 350$, $c = 450$, find R, the radius of the circle circumscribing the triangle by using the relation $4RS = abc$. Find also the angle A. See § 82.

25. If two sides of a triangle are 1026 and 1824 and the difference of the angles opposite these sides is 51° 1.6′, find the remaining parts of the triangle.

HINT. Use law of tangents.

26. The angles of a triangle are 74°, 59°, and 47°. If the perimeter is 180 solve the triangle making use of the formula $r = s \tan A/2 \tan B/2 \tan C/2$, where r is the radius of the inscribed circle and s is half the sum of the sides.

27. Find the perimeter of the triangle if two sides and included angle are 5544, 6691, and 8° 20.2′ respectively.

28. Solve the triangle, given $r = 15$, $s = 80$, $A = 50° 18'$.

HINT. Use a half angle formula and Newton's formula.

29. The area of a triangle is 1234, its angles are 24° 45.6′, 75° 37.5′, and 79° 36.9′. Solve the triangle by using the relation $S = 2R^2 \sin A \sin B \sin C$.

HINT. See Exercise 14, § 47.

30. The area of a triangle is 1000 square feet, and two of its angles are 46° 35.9′ and 68° 7.3′. Use the relation $S = 2R^2 \sin A \sin B \sin C$ to solve the triangle.

31. Find the radius of the largest gasoline storage tank that can be built on a triangular piece of ground whose dimensions are 31.2, 42.3, and 34.2 feet.

32. The radius, r, of the inscribed circle of a triangle is 120 feet and two of the angles of the triangle are 46° 16′ and 79°. Use the relation

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

to solve the triangle.

HINT. See Exercise 7, p. 65.

33. Two angles of a triangle are $75° 36.5'$ and $60°$ and the radius of the circumcircle is 246 feet. Find the length of each side.

34. Two angles of a triangle are $70°$ and $65°$ and the radius of the inscribed circle is 200 inches. Find the sides of the triangle.

35. In Fig. 104 three circles with radii 3, 4, and 5, are tangent externally. Find the angles between the lines joining the centers.

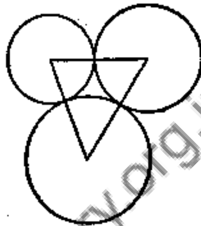


FIG. 104

36. Using the formulas, $\tan A/2 = r/(s - a)$, etc., find the angle between the largest and the smallest sides of the triangle whose sides are $a = 31.7$, $b = 39.1$, $c = 50$.

37. A man standing between two towers, 200 feet from the base of the higher, which is 90 feet high, observes their elevations to be the same; 70 feet nearer the shorter tower he finds the elevation of one is twice that of the other. Find the height of the shorter tower and his original distance from it.

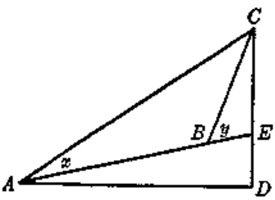


FIG. 105

38. In Fig. 105, given $DAC = 33°$, $x = 20°$, $y = 55°$, and $AB = 1575$ feet; find EC.

39. Prove that the area A of a regular polygon of n sides circumscribed about a circle of radius r is given by the relation

$$A = nr^2 \tan (180°/n).$$

Find the area of a regular polygon of 17 sides circumscribed about a circle of radius 20 feet.

40. Show that the area A of a regular polygon of n sides inscribed in a circle of radius r is given by the relation

$$A = nr^2 \sin (180°/n) \cos (180°/n) = \tfrac{1}{2}nr^2 \sin (360°/n).$$

Find the area of a polygon of 23 sides inscribed in a circle of radius 56.7 feet.

41. Show that the perimeter of a regular inscribed polygon of n sides in a circle of radius r is given by the expression $2nr \sin (180°/n)$. Find the perimeter of the polygon in Exercise 40.

42. Prove that the area of any quadrilateral is one half the product of the two diagonals multiplied by the sine of their included angle.

43. From the first relation below, derive the second:

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}, \qquad \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

HINT. See § 47. Write similar expressions for $\sin (B/2)$ and $\sin (C/2)$.

44. From the first relation below, derive the second:

$$\cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}, \qquad \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}.$$

See § 47. Write similar expressions for $\cos (B/2)$ and $\cos (C/2)$.

45. From the first relation below, show that the other two relations hold for any right triangle:

$$\tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}},\qquad \tan\frac{A}{2}=\sqrt{\frac{c-b}{c+b}},\qquad \tan\frac{B}{2}=\sqrt{\frac{c-a}{c+a}}.$$

46. Express $\sin A$ in terms of the semi-perimeter s and the sides.

HINT. Use $\sin A = 2\sin(A/2)\cos(A/2)$.

47. By using $S = \frac{1}{2}ab\sin C$ for the area of a triangle, and the results in Exercise 46, derive the formula

$$S=\sqrt{s(s-a)(s-b)(s-c)}.$$

48. Using the identity

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

express $\sin A + \sin B + \sin C$ in terms of the sides and semiperimeter s of the triangle ABC.

49. Express R in terms of the sides and semiperimeter s of the triangle ABC.

50. Apply the formulas in Exercise 43 to solve Exercise 22.

51. Apply the formulas in Exercise 44 to solve Exercise 21.

52. If one angle of a triangle is 60°, the area $10\sqrt{3}$ square feet, and the perimeter 20 feet, find the lengths of the sides.

53. The shadow of a tower standing on a level plane is found to be 60 feet longer when the altitude of the sun is 30° than when it is 45°. Find the height of the tower.

54. A square tower stands upon a horizontal plane. From a point in this plane from which three of its upper corners are visible their angular elevations are respectively 45°, 60°, and 45°. Show that the height of the tower is to the breadth of one of its sides as $\sqrt{6}(\sqrt{5}+1)$ is to 4.

55. In any circle, prove that a chord which subtends an angle of 108° at the center is equal to the sum of the two chords which subtend central angles of 36° and 60°.

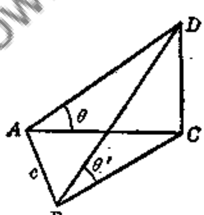


FIG. 106

56. Given $\tan(A/2) = 5/6$ and $\tan(B/2) = 20/37$, find $\tan(C/2)$ and then show that for this triangle $a + c = 2b$.

57. The area of a triangle is 5424, and two of its sides are 106 and 138. Find the angle between them.

58. In Fig. 106, the points A, B, C lie in a horizontal plane. The point D lies directly over the point C. Show that

$$CD=\frac{c\sin A\tan\theta'}{\sin(A+B)}=\frac{c\sin B\tan\theta}{\sin(A+B)}.$$

59. To find the height of a mountain peak above a horizontal plane, the following measurements were made according to the figure in Exercise 58, $c = 1200$, $A = 45°$, $B = 70°\,13'$, $\theta' = 35°$. Find the height of the peak.

60. Find the angle of elevation of the mountain peak in Exercise 59 from the point A.

61. Two cabins A and B are located on the side of a mountain. Call the top of the mountain T and let T' and B' be the feet of the perpendiculars dropped from T and B, respectively, upon the horizontal plane through A. The angles of elevation of T and B from A are 25° and 15°, respectively. Given that T' bears N 20° E from A and N 35° W from B. If B bears S 75° E from A, find the heights of T and B above the horizontal plane containing A, if AB is 1565 yards.

62. Two straight roads intersect at an angle of 30°. From the point of intersection two pedestrians A and B start at the same time, A walking along one road at a rate of 5 miles per hour, and B walking at a uniform rate along the other road. At the end of 3 hours they are 9 miles apart. Find the rate (or rates) at which B walks.

63. Using the formulas in Exercise 45, find b in the right triangle in which $A = 88° 18'$ and $c = 1000$. Complete the solution of the triangle.

CHAPTER X

RADIAN MEASURE OF ANGLES

84. The Radian. In measuring angles, we have thus far used degrees, minutes, and seconds as the units. Another unit, *the radian,* is more convenient for many purposes, notably in certain problems involving the methods of the calculus.

DEFINITION. *A radian is an angle which, if its vertex is placed at the center of a circle, intercepts an arc equal to the radius.* Thus, in Fig. 107, the arc r subtends a radian at O.

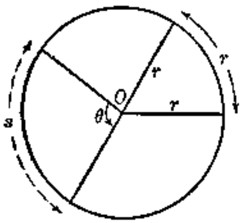


FIG. 107

85. Relation Connecting Arc, Angle, and Radius. In any circle of radius r, let a central angle of θ radians (Fig. 107) intercept on the circle an arc s. Since two angles of a circle have the same ratio as their intercepted arcs (plane geometry), *the radian measure of the angle θ is the ratio of the length of its arc s to the radius r.* That is,

(1) $$\theta = \frac{s}{r}.$$

Solving (1) for s, we get

(2) $$s = r\theta.$$

In another form,

(3) $$\text{arc} = (\text{radius})(\text{angle in radians}).$$

86. Relation between Degrees and Radians. In the special case of a straight angle, the arc s in formula (2), § 85, is a semicircumference. Thus, from geometry, $s = \pi r$. Substituting in (1), § 85, we have

$$\text{straight angle } \theta = \frac{\pi r}{r} = \pi$$

for the number of radians in a straight angle. Thus, as an approximation, we may say roughly that a straight angle is equal to $3\frac{1}{7}$ radians. Since a straight angle contains 180°, it follows that

(4) $$\pi \textbf{ radians} = \mathbf{180°}.$$

This is a fundamental relation for changing radians to degrees and degrees to radians. The relation (4) may be written in other forms that tend to facilitate calculations. Thus, from (4),

$$(5)\qquad 1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.1416}, \text{ approximately.}$$

$$(6)\qquad 1 \text{ radian} = 57.296^\circ = 57^\circ\ 17.8', \text{ approximately.}$$

On the other hand, we find, from (4),

$$(7)\qquad \mathbf{1^\circ = \frac{\pi}{180}\ radians = 0.017453\ radians,\ approximately.}$$

From (5), (6), and (7), we obtain the following rules for changing measurements in radians to degrees and vice versa:

(a) *To change degrees to radians, multiply the number of degrees by*

$$\frac{\pi}{180} = 0.017453 \cdots.$$

(b) *To change radians to degrees, multiply the number of radians by*

$$\frac{180}{\pi} = 57.296 \cdots.$$

Hereafter, when no other unit of measure for angles is indicated, it will be assumed that an angle is expressed in radians. Thus, by $\pi/2$, we mean $\pi/2$ radians which equals 90°.

The number of radians in an angle is frequently expressed in terms of π. Thus, 360°, 180°, 90°, 60°, 45°, 30° are respectively equal to 2π, π, $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$ radians.

EXERCISES

Express the following in radian measure, giving results in terms of π.

1. 120°
2. 135°
3. 150°
4. 210°
5. 225°
6. 240°
7. 270°
8. 300°
9. 330°
10. 315°
11. 60°
12. − 120°
13. − 240°
14. − 20°
15. − 60°
16. − 50°

Change the following radian measures of angles to degrees:

17. $\pi/2$
18. $\pi/4$
19. $\pi/3$
20. $\pi/9$
21. $3\pi/2$
22. $3\pi/4$
23. $2\pi/3$
24. $7\pi/12$
25. $5\pi/6$
26. $5\pi/12$
27. $-\pi$
28. $8\pi/15$
29. 1
30. 3
31. 2.5
32. $-8\pi/9$
33. 4
34. -2π
35. $-\pi/10$
36. $11\pi/6$

37. Through how many radians does the minute hand of a clock revolve in 15 minutes? Through how many degrees?

38. How many radians between the hour and minute hand of a clock at five o'clock? How many degrees?

39. One angle of a triangle is 45°, and another is $\pi/3$ radians. What is the third angle in radians?

40. Express 24° 25′ in radians giving the result (a) exactly in terms of π and (b) approximately in decimals.

SOLUTION. (a) Since

$$24° 25' = 24\frac{25°}{60} = 24\frac{5°}{12} = \frac{293°}{12},$$

we have

$$24° 25' = \frac{293}{12} \cdot \frac{\pi}{180} \text{ radians} = \frac{293\pi}{2160} \text{ radians.}$$

(b) Substituting $\pi = 3.1416$, we obtain

$$24° 25' = \frac{(293)(3.1416)}{2160} \text{ radians} = 0.4262 \text{ radians (nearly).}$$

The result could also be obtained from Table II, page 24, by interpolation between the two values

$$24° 20' = 0.4247 \text{ radians,}$$
$$24° 30' = 0.4276 \text{ radians;}$$

from which we find

$$24° 25' = 0.4262 \text{ radians.}$$

Express the following in radian measure giving the results in decimal form.

41. 120°	43. 42° 20′	45. 69° 25′	47. 61° 55′
42. 24° 16′	44. 58° 45′	46. 79° 45′	48. 183° 18′

Without the use of a table, give the value of each of the following functions.

49. $\sin(\pi/2)$	52. $\cot(\pi/4)$	55. $\tan(3\pi/4)$	58. $\sin(9\pi/2)$
50. $\cos\pi$	53. $\sin(\pi/6)$	56. $\sin(2\pi/3)$	59. $\tan 9\pi$
51. $\tan(\pi/4)$	54. $\cos(\pi/3)$	57. $\cos 7\pi$	60. $\cot(7\pi/2)$

87. An Angle in Radian Measure Compared with the Values of its Trigonometric Functions.

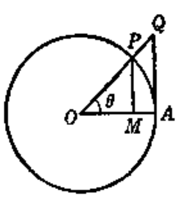


FIG. 108

Both the radian measure of the angle θ and $\sin\theta$ (Fig. 108) are ratios. We can thus compare their values by asking: Which is the larger number, θ in radians or $\sin\theta$, when θ is an acute angle? Answer from Fig. 108 by using the definitions of $\sin\theta$ and the radian measure of θ. Which do you think is the larger number, θ or $\tan\theta$, when θ is an acute angle? See Fig. 108.

When θ is small, θ in radians and $\sin\theta$ are nearly equal as you can verify by examining the values in Table II, page 22. This fact is often used in applied mathematics.

The fact that the angle in radians and the trigonometric functions are both ratios suggests that by taking x in radians and using the same scale for x and y in plotting (see § 40) the equation

$$y = \sin x,$$

we obtain a graphic comparison of the magnitudes of x (in radians) and the value of $\sin x$.

Thus, the length for 90° would be taken as $\pi/2 = 1.57+$ in accord with the suggestion of § 40 that it be taken slightly more than 3/2 of the unit on OY.

88. Area of a Sector. The area bounded by two radii and an arc of a circle is called a *sector*. See AOB of Fig. 109. In geometry it is shown that the area of a sector AOB is equal to arc AB multiplied by one-half the radius. That is,

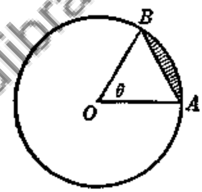


FIG. 109

(8) $$\text{area sector } AOB = \frac{r}{2}(\text{arc } AB) = \frac{r}{2}(r\theta) = \frac{1}{2}r^2\theta.$$

EXAMPLE. The area of the triangle AOB (Fig. 109) is given by the formula $\frac{1}{2}r^2 \sin\theta$ (§ 47), and the area of the sector AOB by formula (8). Show that the area of the shaded segment (Fig. 109) $= \frac{1}{2}r^2(\theta - \sin\theta)$.

EXERCISES

1. Find the length of a circular arc whose radius is 10 inches and which subtends an angle of 1.2 radians.
2. Find the number of radians in an angle at the center of a circle of radius 40 inches which intercepts an arc of 70 inches.
3. The minute hand of a clock is 4 inches long. How far does its end travel in 15 minutes?
4. The radius of a circle is 1000 feet. Find the arc that subtends an angle of 35°.
5. The radius of a circle is 1000 feet. Find the arc that subtends 237° 15′.
6. An angle of 1.5 radians at the center of a circle intercepts an arc of length 105 inches. Find the radius.
7. If we assume the earth to be a sphere of radius 4000 miles, how far is it on the earth's surface from a point in latitude 41° 40′ to the North Pole?

8. A railroad curve is laid out on a circle of radius 1000 feet. What is the length of a portion that subtends an angle of 21° 15′ at the center of the circle?

9. If an arc, 140 feet long, subtends an angle of 4 radians at the center of a circle, find its diameter.

10. If a pendulum 37.5 inches long swings through 0.097 radian, what is the length of the arc through which it swings?

11. Given that θ is the radian measure of a positive acute angle which varies from 0° to 90°. By the use of values in Table II investigate whether $\tan\theta > \theta$, $\tan\theta < \theta$, or $\tan\theta = \theta$.

12. Show that if θ is the radian measure of a positive acute angle, then $\tan\theta > \theta$.

HINT. Use the fact from Fig. 108 that $AQ >$ arc AP.

13. Find the area of a sector of a circle of radius 100 inches if the sector angle is 1.5 radians. If $\pi/2$ radians. If 60°.

14. Find the area of a 25° sector of a circle of radius 200 feet.

15. In a circle of radius 100 feet a chord is 160 feet. Find the area of the smaller segment cut off by the chord.

16. Given a cylindrical oil tank of length 60 feet and radius 8 feet lying horizontally and filled to a depth of 2.5 feet. Calculate the number of gallons of oil in the tank. (Use 1 gal. = 231 cu. in.)

CHAPTER XI

INVERSE TRIGONOMETRIC FUNCTIONS

89. Meaning of an Inverse Trigonometric Function. With a mathematical operation, there is usually associated an inverse operation. For example, with the direct operation of finding the square of a number x, say $y = x^2$, there is associated the inverse operation of finding the number $x = \pm \sqrt{y}$ whose square is y. As another example, with the simple direct process of finding the values of the trigonometric functions of a given angle from a table, there is associated the inverse process of finding the angle from the table when some function of the angle is given. In the equation

$$(1) \qquad x = \sin y$$

we may regard the angle y or arc y as a function of x. This new way of looking at the relation (1) may be expressed by saying

(2) ***y* is an angle whose sine is *x***

or

(3) ***y* is an arc whose sine is *x*.**

The statement (2) or (3) is expressed in the contracted notation

$$(4) \qquad y = \text{arc} \sin x$$

and is read ***arc sine of x***, or the ***inverse sine of x***, or the ***anti-sine of x***.

Another way of writing the equation (4) is

$$y = \sin^{-1} x.$$

The -1 in the symbol $\sin^{-1} x$ should not be confused with an exponent -1. If we wish to write $\sin x$ with an exponent -1, we should write $(\sin x)^{-1}$ and not $\sin^{-1} x$.

The other inverse trigonometric functions, arc $\cos x$, arc $\tan x$, arc $\cot x$, arc $\sec x$, arc $\csc x$, have meanings similar to arc $\sin x$ as will be suggested in the following exercises.

EXERCISES

1. Give meaning to the expression arc cos x.

SOLUTION. Let $x = \cos y$,
then $y = \text{arc} \cos x$,
that is, y is an angle whose cosine is x.

2. Give meanings to the expressions: arc tan x, arc cot x, arc sec x, arc csc x.

3. The equations $x = \sin y$ and $y = \text{arc} \sin x$ are equivalent. Write another equation equivalent to each of the following:

(a) $y = \text{arc} \cos x$.
(b) $x = \tan y$.
(c) $x = \sec z$.
(d) $y = \csc^{-1} x$.
(e) $z = \text{arc} \cot x$.
(f) $u = \sin^{-1} v$.

Write the following in the notation of inverse functions:

4. $\sin y = 1/2$.
5. $\cos y = -1/2$.
6. $\tan y = 2$.
7. $\sec y = -3$.
8. $\csc x = 4$.
9. $\cot \theta = 2$.

Assume a non-negative angle not greater than 90°, and give the value of each angle both in degrees and in radians:

10. arc cos 1
11. arc tan 1
12. $\sin^{-1} (1/2)$
13. arc cos $(\sqrt{2}/2)$
14. arc cot 0
15. arc tan $(1/\sqrt{3})$
16. arc sec $\sqrt{2}$
17. $\sec^{-1} 2$
18. arc sin $(\sqrt{2}/2)$
19. $\tan^{-1} \sqrt{3}$
20. arc tan $(\sqrt{3}/3)$
21. arc sin 0

Simplify:

22. arc sin (sin x)
23. sin (arc sin x)
24. arc tan (tan x)
25. tan (arc tan x)
26. $\cos \cos^{-1} x$
27. arc cos (cos x)

90. Multiple Valuedness of the Inverse Functions. A fundamental difference between the direct and inverse trigonometric functions may be illustrated by the relation between an angle and its sine. While the sine of an angle has only one value, there are many angles that have the same value for the sine. For example, arc sin (1/2) may be 30°, 150°, 390°, $-210°$, $\cdots$. In fact, if A is a value of arc sin x, so is $(180° - A)$ and so is any angle obtained from A or from $(180° - A)$ by adding or subtracting any multiple of 360°.

We thus say that each inverse trigonometric function, such as $y = \text{arc} \sin x$, is a ***multiple-valued*** function, since, corresponding to any value of x for which the function is defined, there are many values of the angle y.

This property of multiple valuedness of the inverse function stands out prominently in a study of the graphs of these functions in § 91.

91. Graphs of the Inverse Functions. Since the two equations

$$x = \sin y \quad \text{and} \quad y = \text{arc} \sin x$$

are equivalent, their graphs are the same.

In Fig. 56, page 57, we have given the graph of $y = \sin x$. If we interchange x and y, we obtain the graph of $y = \text{arc} \sin x$ shown in Fig. 110.

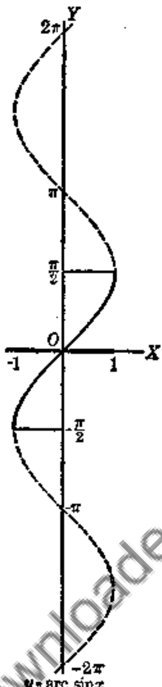


FIG. 110

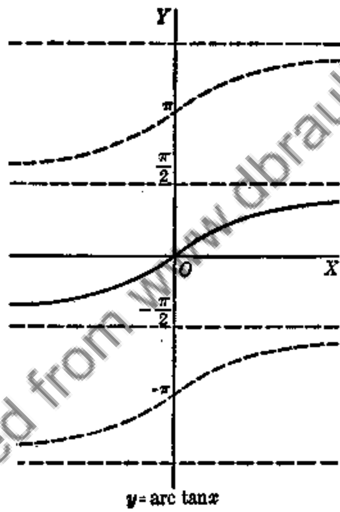


FIG. 111

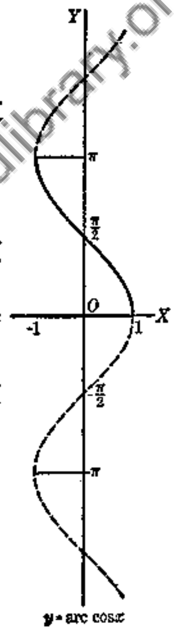


FIG. 112

Similar treatment of the graphs in Figs. 57–61, pp. 58–60, would give the graphs of the remaining inverse functions. The graphs of arc tan x, arc cos x are shown in Figs. 111, 112.

As the graph of each of the inverse functions could be continued for an unlimited distance both upward and downward, we have a picture which shows that for any assigned value of x for which the inverse function is defined, say arc sin (1/2), there are infinitely many values of the angle.

But it should be noted especially that arc sin x and arc cos x are defined only when x is taking values from -1 to $+1$ inclusive, while arc csc x and arc sec x are defined except when x is between -1 and $+1$.

92. Principal Values of Inverse Functions. We have observed in §§ 90 and 91 that if

$$y = \text{arc} \sin x,$$

then infinitely many values of y correspond to each assigned value of x between -1 and $+1$. For certain purposes, one of these values is selected and called the ***principal value***.

We define the ***principal value**** of an inverse trigonometric function to be its smallest numerical value, selecting the positive value when two equally small numerical values are of opposite signs.

Thus the principal value of arc sin $(-1/2)$ is $-30° = -\pi/6$, and of arc sec (-2) is $120° = 2\pi/3$.

For the graphs of the principal values of arc sin x, arc cos x, and arc tan x, see the heavy type portions of graphs in Figs. 110–112. From these heavy type graphs, it may be observed that the principal values of arc sin x and arc tan x range from $-\pi/2$ to $\pi/2$, while those of arc cos x range from 0 to π.

EXERCISES

Give each principal value both in degrees and radians:

1. arc sin (1/2)
2. arc sin $(-1/2)$
3. $\cos^{-1}(-1/2)$
4. arc cos (-1)
5. arc cos $(-\sqrt{3}/2)$
6. arc sin (-1)
7. arc sin $(\sqrt{2}/2)$
8. arc tan (-1)
9. arc cot (-1)
10. arc sin $(-\sqrt{2}/2)$
11. arc sin 0
12. arc cos 0
13. arc tan 0
14. $\tan^{-1}(\frac{1}{3}\sqrt{3})$
15. arc tan $(-1/\sqrt{3})$
16. arc tan 1
17. arc cot $(-\sqrt{3}/3)$
18. arc cot $\sqrt{3}$
19. arc sin 1
20. arc csc 1
21. arc sec 2
22. arc sec (-2)
23. arc sec (-1)
24. arc tan $(-\sqrt{3})$
25. arc cos $(-\sqrt{2}/2)$
26. $\cos^{-1}(\sqrt{3}/2)$
27. arc csc (-1)

Using Table II, find the principal value in degrees and minutes:

28. arc sin 0.2588
29. arc cos 0.9559
30. arc tan (-0.3552)
31. arc cot 2.279

* Authors of books on trigonometry and calculus differ somewhat in their selections of principal values.

32. Sketch the graph of the principal values of each of the following:

$$y = \text{arc cot } x,$$
$$y = \text{arc sec } x,$$
$$y = \text{arc csc } x.$$

93. Practice in Finding Values of Functions when Angles are Given in the Notation of Inverse Functions. The notation of inverse functions is a language that can be mastered by practice on examples.

EXAMPLE 1. Find tan arc sin (3/5), using the principal value of arc sin (3/5).

SOLUTION. Let arc sin $(3/5) = \theta$; then $\sin \theta = 3/5$.

By the limitation to the principal value, θ must be in the first or fourth quadrant, and since the sine is not positive in the fourth quadrant, we know that θ is in the first quadrant. From Fig. 113, $\tan \theta = 3/4$. Hence $\tan \text{arc sin } \frac{3}{5} = \frac{3}{4}$.

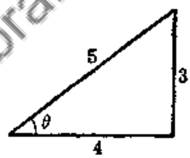


FIG. 113

EXAMPLE 2. Find sin arc cos $(-\frac{1}{2}\sqrt{3})$, using the principal value of arc cos $(-\frac{1}{2}\sqrt{3})$.

SOLUTION. Note that arc cos $(-\frac{1}{2}\sqrt{3}) = 150° = 5\pi/6$. Then

$$\sin \text{arc cos } (-\tfrac{1}{2}\sqrt{3}) = \sin \frac{5\pi}{6} = \frac{1}{2}.$$

EXAMPLE 3. Simplify arc tan $\dfrac{2 \tan x}{1 - \tan^2 x}$.

SOLUTION. From § 52, $\dfrac{2 \tan x}{1 - \tan^2 x} = \tan 2x$.

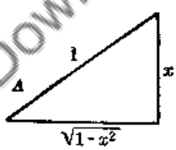


FIG. 114

Hence, we are to find arc tan (tan $2x$) which is obviously $2x$.

EXAMPLE 4. Simplify cos (arc sin x).

SOLUTION. Let $A = \text{arc sin } x$ (Fig. 114). Then $\sin A = x$, and

$$\cos A = \pm \sqrt{1 - \sin^2 A} = \pm \sqrt{1 - x^2}.$$

Thus, $\cos A = \cos (\text{arc sin } x) = \pm \sqrt{1 - x^2}$.

Which sign should be taken before the radical if we were limited to the principal value of arc sin x?

EXAMPLE 5. Simplify $\sin(\text{arc}\sin x + \text{arc}\cos x)$, using only principal values.

SOLUTION. Let $A = \text{arc}\sin x$, $B = \text{arc}\cos x$; then $\sin A = x$ and $\cos B = x$,

$$\begin{aligned}\sin(\text{arc}\sin x + \text{arc}\cos x) &= \sin(A + B)\\ &= \sin A \cos B + \cos A \sin B\\ &= x^2 + \sqrt{1 - x^2}\sqrt{1 - x^2} = 1.\end{aligned}$$

EXERCISES

Find the value of each of the expressions, using the principal value of the angle:

1. $\tan \text{arc}\sin(\sqrt{3}/2)$
2. $\tan \text{arc}\cos(-1/2)$
3. $\sin \text{arc}\tan(-1)$
4. $\tan \sin^{-1}(-1/2)$
5. $\csc \text{arc}\tan 1$
6. $\sin \text{arc}\cos(-\sqrt{3}/2)$
7. $\sin \text{arc}\tan(-12/5)$
8. $\sec \text{arc}\cot(\sqrt{3})$
9. $\tan \text{arc}\sin(-\sqrt{2}/2)$
10. $\tan(180° - \sin^{-1} 5/13)$
11. $\cos \tan^{-1}(-4/3)$
12. $\sin \sec^{-1}(-13/12)$
13. $\cos \cot^{-1}(-12/5)$
14. $\sin[\pi/2 + \cos^{-1}(3/5)]$
15. $\sin[\pi/2 - \sin^{-1}(1/2)]$
16. $\tan[\pi/2 - \text{arc}\tan(\sqrt{3}/3)]$
17. $\cot[270° - \text{arc}\tan(-2/3)]$
18. $\sin[\pi - \sin^{-1}(1/3)]$

Simplify each of the expressions, using only principal values in Exercises 19, 22, 23, 25, 28.

19. $\sin(2\,\text{arc}\sin x)$
20. $\cos(2\,\text{arc}\cos x)$
21. $\tan(2\tan^{-1} x)$
22. $\cos(\text{arc}\sin x + \text{arc}\cos x)$
23. $\cos(\text{arc}\sin x + \text{arc}\cos y)$
24. $\text{arc}\sin(2\sin x\cos x)$
25. $\sin(\text{arc}\sin x - \text{arc}\sin y)$
26. $\tan(\text{arc}\tan x + \text{arc}\tan y)$
27. $\tan(\cot^{-1} x - \cot^{-1} y)$
28. $\cos(\sin^{-1} x - \sin^{-1} y)$
29. $\sin^2(\frac{1}{2}\text{arc}\cos x)$
30. $\cos^2(\frac{1}{2}\text{arc}\cos x)$

94. Identities Involving Inverse Functions. Identities involving inverse functions can be best studied from simple examples. It will be found especially important to indicate the nature of the restrictions on the variables and inverse functions in these examples.

EXAMPLE 1. Show that the relation

$$\text{arc}\tan x + \text{arc}\cot x = \pi/2$$

is satisfied by an infinite number of values of x. Discuss the restrictions on $\text{arc}\tan x$ and $\text{arc}\cot x$ such that they will satisfy the given relation.

SOLUTION. From Fig. 115, we note that

$$A + B = 90° = \pi/2$$

or

(1) $$\text{arc tan } x + \text{arc cot } x = \pi/2.$$

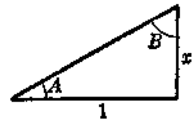


FIG. 115

The relation (1) is thus obviously satisfied if we restrict arc tan x and arc cot x to any one of the infinite set of angles from 0 to $\pi/2$. Moreover, it is likewise easily seen from the geometry of similar triangles that (1) is satisfied when arc tan x is in the interval $-\pi/2$ and 0, and arc cot x has the corresponding value in the interval $\pi/2$ to π.

While (1) is thus satisfied for these angles, it is not satisfied for arbitrarily selected values of arc tan x and arc cot x that correspond to an assigned x as may be seen by taking $x = -1$, and certain corresponding values of the inverse functions, say arc tan $x = 7\pi/4$ and arc cot $x = 3\pi/4$.

EXAMPLE 2. Show that

$$\text{arc tan } x + \text{arc tan } y = \text{arc tan } \frac{x + y}{1 - xy}.$$

SOLUTION. Let arc tan $x = A$, and arc tan $y = B$. That is,

$$\tan A = x, \qquad \tan B = y.$$

Then

$$\tan (A + B) = \frac{x + y}{1 - xy}.$$

That is,

$$A + B = \text{arc tan } \frac{x + y}{1 - xy},$$

or

$$\text{arc tan } x + \text{arc tan } y = \text{arc tan } \frac{x + y}{1 - xy}.$$

EXAMPLE 3. Show that $\sin^{-1} x = \tan^{-1} \dfrac{x}{\sqrt{1 - x^2}}$ when the values of the inverse functions are principal values.

SOLUTION. Let $\sin^{-1} x = A$; that is, $\sin A = x$. Then

$$\tan A = \frac{\sin A}{\cos A} = \frac{x}{\sqrt{1 - x^2}}$$

where $\sqrt{1 - x^2}$ is used rather than $-\sqrt{1 - x^2}$ since the angle A is restricted to a principal value. The identity is established because the tangent of the angle in the left member has been proved equal to the tangent of the angle in the right member.

EXERCISES

Verify each of the following statements.

1. $\cos^2 \text{arc} \sin x = 1 - x^2$.
2. $\sec^2 \text{arc} \tan x = 1 + x^2$.
3. $\cot \text{arc} \tan x = 1/x$.
4. $\csc \text{arc} \sin x = 1/x$.
5. $\tan (2 \text{ arc} \tan x) = 2x/(1 - x^2)$.
6. $\cos (2 \text{ arc} \cos x) = 2x^2 - 1$.

By restricting the inverse functions to principal values, verify the following statements.

7. $\sin^{-1} x + \cos^{-1} x = \pi/2$.
8. $\sin (\text{arc} \sin x + \text{arc} \sin y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.
9. $\sin \text{arc} \cos x = \sqrt{1 - x^2}$.
10. $\text{arc} \tan x = \text{arc} \sin (x/\sqrt{1 + x^2})$.

CHAPTER XII

TRIGONOMETRIC EQUATIONS

95. Introduction. The equality

(1) $$2 \sin x + 1 = 0$$

is not satisfied by $x = 0$, nor by any other angle between 0° and 360°, excepting the two angles 210° and 330°. It is therefore not an identity (see § 34).

An equality that is not an identity and that contains a trigonometric function of an *unknown* angle is called a ***trigonometric equation.*** Thus, the equality (1) is an example of a trigonometric equation.

EXERCISES

Which of the following equalities are trigonometric equations?

1. $\sin x + 2 \cos x = 1.$
2. $\sin^2 x + \cos^2 x = 1.$
3. $\sec^2 x - 2 = 0.$
4. $2x \sin 40° - 7x \tan 20° + 5 = 0.$

96. Solution of a Trigonometric Equation. If a trigonometric equation in x has a solution $x = A$, solutions of the form $A + 2n\pi$ (n an integer) frequently occur on account of the periodicity (§ 40) of the functions. For example, we get solutions of (1), § 95, when we add a multiple of 360°. Thus, since the period of $\sin x$ is 360°, all the solutions of $2 \sin x + 1 = 0$ may be written in the form

$$x = 210° + n \cdot 360°, \qquad x = 330° + n \cdot 360°,$$

where n is zero or any integer, positive or negative. In what follows, n will be used with this meaning. In radian measure, these solutions may be written in the forms

$$7\pi/6 + 2n\pi, \quad \text{and} \quad 11\pi/6 + 2n\pi.$$

No attempt will be made to give general rules for solving trigonometric equations. The illustrative examples give methods that may often be employed.

EXAMPLE 1. Solve $2 \sin x - 1 = 0.$

SOLUTION. We find at once that $\sin x = \frac{1}{2}$. The angles whose sine is $\frac{1}{2}$ are $30° + n \cdot 360°$, and $150° + n \cdot 360°$. In radian measure, these angles are $\pi/6 + 2n\pi$, and $5\pi/6 + 2n\pi$.

EXAMPLE 2. Solve $\tan x - 0.3542 = 0$.

SOLUTION. We find at once that $\tan x = 0.3542$. From Table II, the acute angle $x = 19° 30'$. Since $\tan(180° + x) = \tan x$, the period of $\tan x$ (§ 40) is 180°. The solutions may be written in the form $x = 19° 30' + n \cdot 180°$.

EXAMPLE 3. In the equation

$$\sec^2 x - \sec x - 2 = 0,$$

find values of x between 0 and 2π radians.

SOLUTION. $$\sec^2 x - \sec x - 2 = 0.$$
Factoring, we have

$$(\sec x + 1)(\sec x - 2) = 0.$$

Then

$$\sec x + 1 = 0, \quad \text{or} \quad \sec x = -1;$$
$$\sec x - 2 = 0, \quad \text{or} \quad \sec x = 2.$$

When $\sec x = -1$, $\quad x = 180° = \pi$ radians.
When $\sec x = 2$, $\quad x = 60°$, or $300°$
$= \pi/3$, or $5\pi/3$ radians.

These are the only solutions between 0 and 360°. As before, other solutions may be found by adding or subtracting any multiple of 360°.

EXAMPLE 4. Given $2\cos^2 x + 3\cos x - 2 = 0$, find x in radians.

SOLUTION. $$2\cos^2 x + 3\cos x - 2 = 0.$$
Factoring, we have

$$(2\cos x - 1)(\cos x + 2) = 0.$$

Then

$$2\cos x - 1 = 0, \quad \text{or} \quad \cos x = 1/2;$$
$$\cos x + 2 = 0, \quad \text{or} \quad \cos x = -2.$$

There is no real angle whose cosine is -2. From $\cos x = \frac{1}{2}$, we have $x = \pi/3 + 2n\pi$, or $5\pi/3 + 2n\pi$. Test whether each of the values satisfies the equation to be solved.

EXERCISES

In each of the following equations find values of x in the range 0° to 360° (excluding 360°).

1. $\tan x + 1 = 0.$
2. $4 \sin^2 x - 1 = 0.$
3. $\tan^2 x - 3 = 0.$
4. $\csc^2 x - 2 = 0.$

Find values of x in the range 0° to 180° (including 180°).

5. $\sin x - 0.4823 = 0.$
6. $2 \sin^2 x - \sin x - 1 = 0.$
7. $\cos x - 0.7660 = 0.$
8. $\sqrt{3} \tan x + \tan x = 0.$

Find both in degrees and in radians, the solutions of each of the following equations.

9. $\cot^2 x - \cot x = 0.$
10. $2 \sin^2 x + \sin x - 1 = 0.$

97. Equations Solved by Use of Identities. In solving a trigonometric equation that contains more than one function of the unknown x, sometimes one or more of the fundamental identities, p. 48, will serve to reduce the equation to a solvable form that contains only one function of x.

In solving trigonometric equations, the student is urged to check solutions by substitution in the given equation. See Example 3, below.

Example 1. Solve the equation $2 \sin^2 x - 3 \cos x = 0.$

Solution.

$$2 \sin^2 x - 3 \cos x = 0.$$

Substituting $\sin^2 x = 1 - \cos^2 x$, we find

$$2(1 - \cos^2 x) - 3 \cos x = 0,$$

that is,

$$2 \cos^2 x + 3 \cos x - 2 = 0.$$

Compare Example 4, § 96, and complete the solution. Check the solutions by substitution in given equation.

Example 2. In the equation $\tan x + 2 \cot x = 6$, find the values of x between 0° and 360°.

Solution.

$$\tan x + 2 \cot x = 6.$$

Substituting $\cot x = 1/\tan x$, we find

$$\tan^2 x - 6 \tan x + 2 = 0.$$

Solving this quadratic for tan x, we find

$$\tan x = \frac{6 \pm \sqrt{36 - 8}}{2} = 3 \pm \sqrt{7} = 5.6458, \quad \text{or} \quad 0.3542$$

From Table II, arc tan 5.6458 = 79° 57′ (acute angle),
arc tan 0.3542 = 19° 30′ (acute angle).

The solutions between 0° and 360° are 79° 57′, 259° 57′, 19° 30′, 199° 30′.

By substitution in the original equation, $\tan x + 2 \cot x = 6$, it is found to be *approximately* satisfied by each of the four values of x. We say *approximately* satisfied, since each of the four values is an approximation.

EXAMPLE 3. Solve $1 + \sin x = \cos x$ for values of x in the range 0° to 360° excluding 360°.

SOLUTION. Substituting $\cos x = \pm\sqrt{1 - \sin^2 x}$, we find

$$1 + \sin x = \pm\sqrt{1 - \sin^2 x}.$$

Squaring both sides, we have

$$1 + 2 \sin x + \sin^2 x = 1 - \sin^2 x,$$
$$2 \sin x + 2 \sin^2 x = 0,$$
$$2 \sin x(1 + \sin x) = 0.$$

Hence $x = 0°, 180°$, or $270°$.

When we test these solutions by substitution in the original equation, we find that 0° and 270° satisfy it, but 180° does not. Hence, 180° is not a solution of the equation to be solved. This example illustrates the importance of testing the validity of results by substitution.

EXAMPLE 4. Find in radians the solutions of

$$2 \cos 2x = 1 \text{ between } 0 \text{ and } 2\pi.$$

SOLUTION. We have $\cos 2x = \frac{1}{2}$. Hence

$$2x = \frac{\pi}{3} + 2n\pi, \quad \frac{5\pi}{3} + 2n\pi.$$

Dividing by 2, we have

$$x = \frac{\pi}{6} + n\pi, \quad \frac{5\pi}{6} + n\pi.$$

By making $n = 0$, and $n = 1$, we obtain

$$\frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{7\pi}{6}, \quad \frac{11\pi}{6}.$$

EXAMPLE 5. Solve

$$\sin x = \cos 2x - 1.$$

SOLUTION. Substituting (see § 52)

$$\cos 2x = 1 - 2\sin^2 x,$$

we find

$$\sin x = -2\sin^2 x,$$
$$2\sin^2 x + \sin x = 0.$$

Factoring, we obtain

$$\sin x(2\sin x + 1) = 0.$$

Hence we have now to solve the two equations

$$\sin x = 0, \qquad 2\sin x + 1 = 0.$$

Both factors give solutions. The solutions of $\sin x = 0$ are $n\pi$. For solutions of $2\sin x + 1 = 0$, see § 95.

EXAMPLE 6. Solve $3\sin x + 4\cos x = 1$ for value of x in the second quadrant.

SOLUTION. Dividing through by $\sqrt{3^2 + 4^2} = 5$, we have

$$\frac{3}{5}\sin x + \frac{4}{5}\cos x = \frac{1}{5}.$$

Put $3/5 = \sin y$. Then $\cos y = 4/5$, so that the equation becomes

$$\sin y \sin x + \cos y \cos x = \frac{1}{5},$$

or

$$\cos(x - y) = \frac{1}{5} = 0.2$$

$$x - y = \text{arc} \cos 0.2 = 78° \, 28' \quad \text{(acute)}.$$

But $y = \text{arc} \cos 4/5 = \text{arc} \cos 0.8 = 36° \, 52'$, approximately. Hence

$$x = y + 78° \, 28' = 36° \, 52' + 78° \, 28' = 115° \, 20', \text{ approximately.}$$

Test this result by substitution in the original equation. Solve Example 6 by substituting $\sin x = \sqrt{1 - \cos^2 x}$ in the given equation.

EXAMPLE 7. Solve arc sin $(x - 2) = \pi/6$ radians.

SOLUTION. Changing to the direct function, we have

$$x - 2 = \sin\frac{\pi}{6} = \frac{1}{2}.$$

Hence $x = 2\frac{1}{2}$.

EXERCISES

Solve the following equations for angles from 0° to 360°, including 0° and excluding 360°.

1. $4\cos^2 x = 3$.
2. $\sin x = 2\cos^2 x - 1$.
3. $\cot^2 x = 1 + \csc x$.
4. $3\cos x = 2\sin^2 x$.
5. $\csc^2 x + \cot x - 3 = 0$.
6. $\tan x + \cot x - 2 = 0$.

Find solutions such that $0° \leqq x \leqq 180°$, and express the answers in radians.

7. $\sin x = 2\sin^2 x$.
8. $\cos 2x(1 + 2\cos x) = 0$.

Find solutions such that $-90° \leqq x \leqq 90°$ and express the answers to Exercises 10 and 11 in radians.

9. $\sin x - 2\cos x + 1 = 0$.
10. $\sin x + \cos x = 0$.
11. $\sin 2x \sin x - \sin 2x \cos x = 0$.

Find the solutions of the following both in degrees and in radians.

12. $\tan^2 x = 3$.
13. $\sin x = 2\cos x$.
14. $4\cos^2 x + 4\sin x - 5 = 0$.
15. $\sin 2x - \cos 2x + 1 = 0$.

Find the solutions for r and θ ($r > 0$, $0° \leqq \theta < 360°$) of the following systems of simultaneous equations:

16. $r = a\cos\theta$, $r\cos\theta = a$. $a > 0$.
17. $r = 6\cos\theta$, $r = \dfrac{3}{2\cos\theta}$.
18. $r\cos\theta = 3$, $r = 4(1 + \cos\theta)$.
19. $r\sin\theta = 3$, $r\cos\theta = 4$.

20. Solve the equation 3 arc tan $x = \pi/4$.

21. Find graphically the approximate solution of the equations

$$y = 2x, \qquad y = \tan x$$

between 0 and $\pi/2$.

22. Find the solutions of

$$y = \sin x, \qquad y = \cos x \cot x$$

between 0 and 2π.

SPHERICAL TRIGONOMETRY

CHAPTER XIII

RIGHT SPHERICAL TRIANGLES

98. Geometrical Introduction. The following facts from the geometry of the sphere will be needed.

(1) A plane section of a sphere is a circle.

(2) A ***great circle*** is the intersection of a sphere with a plane through the center of the sphere.

(3) The ***poles of a great circle*** are the points in which the sphere is cut by the line through the center of the sphere perpendicular to the plane of the great circle.

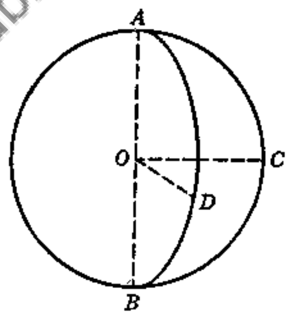


FIG. 116

(4) Two great circles divide the sphere into four parts, each of which is called a ***lune***. Thus, $ACBDA$ (Fig. 116) is a lune.

(5) Any angle formed by two great circles is measured by the dihedral angle between the planes of the great circles forming the sides of the lune. Thus, COD measures each of the two equal angles A and B of the lune.

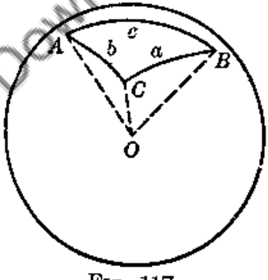


FIG. 117

(6) A ***spherical triangle*** is the portion of a spherical surface bounded by three arcs of great circles meeting in three points. Thus, ABC (Fig. 117) is a spherical triangle. The angles of a spherical triangle are usually denoted by A, B, C, the sides opposite by a, b, c, respectively, as in the case of a plane triangle.

(7) The sides of a spherical triangle as well as the angles will be measured in degrees. Thus, the center O of the sphere is the apex of a

trihedral angle $O\text{–}ABC$. The sides a, b, c of the spherical triangle (Fig. 117), measured in degrees, are equal to the corresponding face angles BOC, COA, AOB, of the trihedral angle.

(8) Although spherical triangles may easily be constructed with one or more sides or angles greater than 180°, we shall consider only those with sides and angles each less than 180°.

(9) The sum of the sides of a spherical triangle is less than 360°.

(10) The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

99. The Polar Triangle. By taking each of the vertices of the spherical triangle ABC as a pole and constructing a great circle, we obtain a set of spherical triangles.

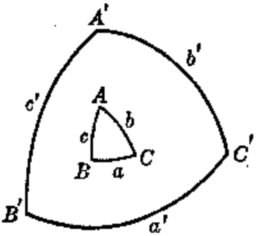


FIG. 118

One of them, $A'B'C'$, (Fig. 118) has the property that A and A' lie on the same side of a; B and B' on the same side of b; and C and C' on the same side of c. This spherical triangle, $A'B'C'$, is then called the ***polar triangle*** of ABC.

In solid geometry, it is proved that if $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$, and that each angle of a spherical triangle is the supplement of the corresponding side of the polar triangle. Thus, from Fig. 118, we write

$$A = 180° - a', \qquad B = 180° - b', \qquad C = 180° - c',$$
$$a = 180° - A', \qquad b = 180° - B', \qquad c = 180° - C'.$$

100. Right Spherical Triangles. The triangle ABC, Fig. 119, on the surface of a sphere is called a ***right spherical triangle*** if one angle, say C, is 90°. Through any point A' of OA, pass a plane perpendicular to OA, intersecting the faces of trihedral angle $O\text{–}ABC$ in $A'B'$, $B'C'$, $C'A'$. The plane angle $C'A'B'$ measures the angle A of the spherical triangle. If the plane OCB is vertical, and the plane OCA is

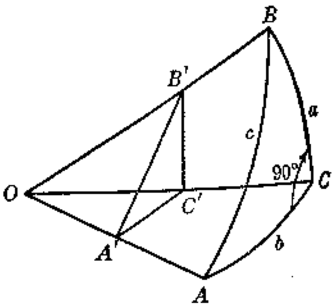


FIG. 119

horizontal (the figure can always be so drawn) it follows that the plane $A'B'C'$ is vertical, and that the angles $OC'B'$, $OA'B'$, $OA'C'$, and $A'C'B'$ are right angles. Then

$$\sin A = \sin C'A'B' = \frac{C'B'}{A'B'} = \frac{\frac{C'B'}{OB'}}{\frac{A'B'}{OB'}} = \frac{\sin a}{\sin c};$$

$$\cos A = \cos C'A'B' = \frac{A'C'}{A'B'} = \frac{\frac{A'C'}{OA'}}{\frac{A'B'}{OA'}} = \frac{\tan b}{\tan c};$$

$$\tan A = \tan C'A'B' = \frac{C'B'}{A'C'} = \frac{\frac{C'B'}{OC'}}{\frac{A'C'}{OC'}} = \frac{\tan a}{\sin b}.$$

Cleared of fractions, these three formulas become:

(1) $$\sin a = \sin c \sin A,$$

(2) $$\tan b = \tan c \cos A,$$

(3) $$\tan a = \sin b \tan A.$$

By drawing a plane perpendicular to OB, and proceeding in the same manner as above, we find three other formulas:

(4) $$\sin b = \sin c \sin B,$$

(5) $$\tan a = \tan c \cos B,$$

(6) $$\tan b = \sin a \tan B.$$

It should be noticed that formulas (4), (5), (6) can be obtained from formulas (1), (2), (3) by interchanging simultaneously the angles A and B, and the corresponding sides a and b. From these six formulas four others may be deduced. Eliminate A from (1), (2), (3) by dividing (1) by (2) and then substituting the value of $\tan A$ thus obtained for $\tan A$ in (3). This gives, after simplification,

(7) $$\cos c = \cos a \cos b.$$

Since $\cot A = 1/\tan A$, it follows from (3), (6), and (7) that

(8) $$\cos c = \cot A \cot B.$$

By using the values of $\cos A$ and $\sin B$ from (2), (4), and (7), it follows that

(9) $$\cos A = \cos a \sin B.$$

By using the values of $\sin A$ and $\cos B$ from (1), (5), and (7), it follows that

(10) $$\cos B = \cos b \sin A.$$

The ten formulas above have been derived from the triangle ABC (Fig. 119), which has each part less than 90° except the right angle C, but by drawing other figures they may be shown to hold for any right triangle whose sides and angles are each less than 180°.

In the numerical solution (§ 102) of right spherical triangles by means of the formulas (1)–(10), the following statements will be found useful.

I. If $c < 90°$, a and b are either both less or both greater than 90°. This follows readily from (7) by use of the fact stated above that each of the sides a and b is less than 180°.

II. If $c > 90°$, one of the other sides is less and the other greater than 90°.

III. A and a are either both less or both greater than 90°, and the same is true of B and b. This follows from (3) and (6).

101. Napier's Rules. The ten formulas (1) to (10) of § 100 may be collected into two simple rules formulated by John Napier, the inventor of logarithms.

In any right spherical triangle (Fig. 120), we call a and b,

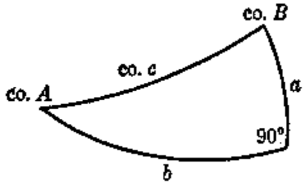


Fig. 120

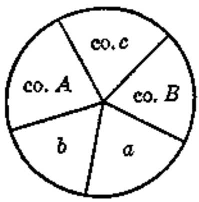


Fig. 121

and the complements of A, c, and B (written co. A, co. c, co. B) the *five circular parts*. The five parts are marked as shown in

Fig. 120, the right angle C being unmarked. If we select any one of the five parts, it has two adjacent parts and two opposite parts (Fig. 121).

Napier's rules are:

(1) The sine of any circular part equals the product of the *tangents* of the *adjacent* parts—(the "tan-adj." rule).

(2) The sine of any circular part equals the product of the *cosines* of the *opposite parts*—(the "cos-opp." rule).

EXERCISE. Prove the correctness of Napier's rules by applying them to each of the five parts in turn, and then reducing the result to one of the formulas (1) to (10), § 100.

For example, if a be taken as the middle part, b and co. B are the adjacent parts, and co. A and co. c are the opposite parts. Then Napier's first rule gives

$$\sin a = \tan b \tan (\text{co. } B) = \tan b \cot B,$$

which is equivalent to formula (6), § 100. Napier's second rule gives

$$\sin a = \cos (\text{co. } A) \cos (\text{co. } c) = \sin A \sin c,$$

which is formula (1), § 100.

102. Solution of Right Spherical Triangles. Napier's rules or the formulas of § 100 are sufficient to solve a right spherical triangle with two parts given besides $C = 90°$. We cannot, however, assign the two parts arbitrarily or we might violate such conditions as (9) or (10), § 98. Indeed, after the solution is carried out, special attention should be given also to conditions I, II, and III (§ 100).

(a) *Solution by use of Napier's rules.* The solution of right spherical triangles by first using Napier's rules to get the appropriate formulas is illustrated in the following examples.

EXAMPLE 1. Given $a = 25° 12.8'$, $B = 71° 36'$, find A, b, and c, using Napier's rules.

SOLUTION.

To find A:	*To find c:*
$\sin (\text{co. } A) = \cos (\text{co. } B) \cos a$,	$\sin (\text{co. } B) = \tan a \tan (\text{co. } c)$,
or $\cos A = \sin B \cos a$,	or $\cot c = \cos B / \tan a$,
$\log \sin B = 9.97721 - 10$,	$\log \cos B = 9.49920 - 10$,
$\log \cos a = 9.95652 - 10$,	$\log \tan a = 9.67288 - 10$,
$\log \cos A = 9.93373 - 10$,	$\log \cot c = 9.82632 - 10$,
$A = 30° 51.2'$.	$c = 56° 9.8'$.

To find b:

$$\begin{aligned} \sin a &= \tan(\text{co. } B)\tan b,\\ \tan b &= \sin a/\cot B,\\ \log \sin a &= 9.62940 - 10,\\ \log \cot B &= \underline{9.52200 - 10},\\ \log \tan b &= 10.10740,\\ b &= 52^\circ\, 0.8'. \end{aligned}$$

Partial Check:

$$\begin{aligned} \sin(\text{co. } A) &= \tan b \tan(\text{co. } c),\\ \cos A &= \tan b \cot c,\\ \log \tan b &= 10.10740,\\ \log \cot c &= \underline{9.82632 - 10},\\ \log \cos A &= 9.93372 - 10. \end{aligned}$$

EXAMPLE 2. (*Ambiguous Case.*) Given $b = 26^\circ\, 4'$ and $B = 36^\circ$, find c, A, and a.

SOLUTION.

To find c:

$$\begin{aligned} \sin b &= \cos(\text{co. } B)\cos(\text{co. } c),\\ \sin c &= \sin b/\sin B,\\ \log \sin b &= 9.64288 - 10,\\ \log \sin B &= \underline{9.76922 - 10},\\ \log \sin c &= 9.87366 - 10,\\ c &= 48^\circ\, 22.9' \text{ or } 131^\circ\, 37.1'. \end{aligned}$$

To find A:

$$\begin{aligned} \sin(\text{co. } B) &= \cos(\text{co. } A)\cos b,\\ \sin A &= \cos B/\cos b,\\ \log \cos B &= 9.90796 - 10,\\ \log \cos b &= \underline{9.95341 - 10},\\ \log \sin A &= 9.95455 - 10,\\ A &= 64^\circ\, 14.5' \text{ or } 115^\circ\, 45.5'. \end{aligned}$$

To find a:

$$\begin{aligned} \sin a &= \tan(\text{co. } B)\tan b\\ &= \cot B \tan b,\\ \log \cot B &= 10.13874 - 10,\\ \log \tan b &= \underline{9.68946 - 10},\\ \log \sin a &= 9.82820 - 10,\\ a &= 42^\circ\, 19.3' \text{ or } 137^\circ\, 40.7'. \end{aligned}$$

Partial Check:

$$\begin{aligned} \sin a &= \cos(\text{co. } A)\cos(\text{co. } c)\\ &= \sin A \sin c,\\ \log \sin A &= 9.95455 - 10,\\ \log \sin c &= \underline{9.87366 - 10},\\ \log \sin a &= 9.82821 - 10. \end{aligned}$$

Since, in this case, each of the unknown parts is determined by the sine, and since we are restricting each part to be less than 180°, we are led to two values for each unknown part. But it does not follow that these values can be taken in every possible way. Indeed, the triangle has just two solutions. An application of I, II, and III, § 100, limits the results to the following values:

$$c = 48^\circ\, 22.9', \quad A = 64^\circ\, 14.5', \quad a = 42^\circ\, 19.3',$$

or

$$c' = 131^\circ\, 37.1', \quad A' = 115^\circ\, 45.5', \quad a' = 137^\circ\, 40.7'.$$

NOTE. If a part is determined from the cosine, tangent, or cotangent, there is only one value for this part.

EXERCISES

Solve each of the following right spherical triangles, using Napier's rules:

1. $A = 50° 0.8'$, $B = 79° 57.3'$
2. $a = 75° 5.3'$, $B = 35° 29.6'$
3. $a = 61°$, $B = 123° 40.2'$
4. $A = 30° 57'$, $B = 69° 26'$

(b) *Solution without the use of Napier's rules.* A right spherical triangle with two parts given besides $C = 90°$ can be solved by using the appropriate formulas selected from formulas (1) to (10), § 100, without deriving them from Napier's rules. There are ten cases but only six that are essentially different.

The following table indicates the given parts, the parts to be found, the formulas of § 100 to be used in each case, and a corresponding check formula.

CASE	GIVEN PARTS	PARTS TO BE FOUND	FORMULAS	CHECK
I	a, b	c, A, B	(7), (3), (6)	(8)
II	a, c	b, A, B	(7), (1),* (5)	(10)
	b, c	a, A, B	(7), (2), (4) *	(9)
III	A, B	a, b, c	(9), (10), (8)	(7)
IV	c, A	a, b, B	(1),* (2), (8)	(6)
	c, B	a, b, A	(5), (4),* (8)	(3)
V	b, A	c, a, B	(2), (3), (10)	(5)
	a, B	b, c, A	(6), (5), (9)	(2)
VI	b, B	a, c, A	(6),* (4),* (10) *	(1)
	a, A	b, c, B	(3),* (1),* (9) *	(4)

As noted on page 148, there may be two possible solutions of the triangle when a part is determined from the sine. In the above table, the formulas that are starred are those from which an unknown part is determined from the sine. In other words, it is in Cases II, IV, and VI in which ambiguities arise. In each of these cases, the situation may be further analyzed with respect to the sizes of the given angles and sides in relation to 90° with the result that there may be two solutions, one solution, no solution, or an infinite number of solutions.

EXAMPLE 1. Given $a = 119° 15'$, $B = 29° 27'$, to find A, b, c by use of formulas selected from (1) to (10), § 100.

SOLUTION.

Use formula (9), $\cos A = \cos a \sin B$.

$$\begin{aligned} \log \cos 119° 15' &= 9.68897_n - 10,^* \\ \log \sin 29° 27' &= 9.69167 - 10, \\ \hline \log \cos A &= 9.38064_n - 10, \\ A &= 103° 54'. \end{aligned}$$

Use formula (6), $\tan b = \sin a \tan B$.

$$\begin{aligned} \log \sin 119° 15' &= 9.94076 - 10, \\ \log \tan 29° 27' &= 9.75176 - 10, \\ \hline \log \tan b &= 9.69252 - 10, \\ b &= 26° 13.6'. \end{aligned}$$

Use formula (5), $\tan c = \dfrac{\tan a}{\cos B}$.

$$\begin{aligned} \log \tan 119° 15' &= 10.25179_n, \\ \log \cos 29° 27' &= 9.93991 - 10, \\ \hline \log \tan c &= 10.31188_n, \\ c &= 115° 59.8'. \end{aligned}$$

Checking results by the use of formula (2), we have

$$\begin{aligned} \tan b &= \tan c \cos A, \\ \log \tan c &= 10.31188_n - 10, \\ \log \cos A &= 9.38064_n - 10, \\ \hline \log \tan b &= 9.69252 - 10, \end{aligned}$$

which checks with the value found above by formula (6).

EXAMPLE 2. Given $b = 66° 46'$, $B = 74° 18'$, find a, c, A by the use of formulas selected from (1) to (10), § 100.

SOLUTION. Use formulas (6), (4), and (10), written

$$\sin a = \frac{\tan b}{\tan B}, \quad \sin c = \frac{\sin b}{\sin B}, \quad \sin A = \frac{\cos B}{\cos b}, \quad \text{respectively.}$$

* The subscript n is used to indicate that $\cos 119° 15'$ is a negative number and that we really show $\log(-\cos 119° 15')$.

$\log \tan b = 10.36725$*	$\log \sin b = 9.96327$	$\log \cos B = 9.43233$
$\log \tan B = \underline{10.55116}$	$\log \sin B = \underline{9.98349}$	$\log \cos b = \underline{9.59602}$
$\log \sin a = 9.81609$	$\log \sin c = 9.97978$	$\log \sin A = 9.83631$
$a_1 = 40° \, 54.1'$	$c_1 = 72° \, 39'$	$A_1 = 43° \, 18.8'$
$a_2 = 139° \, 5.9'$	$c_2 = 107° \, 21'$	$A_2 = 136° \, 41.2'$

Of the eight possible combinations of two solutions for each of the values a, c, A, the admissible solutions under conditions I, II, and III, § 100, are determined as follows.

When $c = 72° \, 39'$, I admits only of $a_1 = 40° \, 54.1'$, and III admits only of $A_1 = 43° \, 18.8'$.

When $c = 107° \, 21'$, II admits only of $a_2 = 139° \, 5.9'$, and III admits only of $A_2 = 136° \, 41.2'$.

Checking results by the use of formula (1), we have

$$\begin{aligned} \sin a &= \sin c \sin A, \\ \log \sin c &= 9.97978 - 10, \\ \log \sin A &= \underline{9.83631 - 10}, \\ \log \sin a &= 9.81609 - 10, \end{aligned}$$

which checks with the above result for $\log \sin a$.

EXERCISES

Solve the following right spherical triangles by the use of formulas (1) to (10), § 100, and the table on page 149.

1. $a = 39° \, 6.8'$, $b = 82° \, 39.6'$
2. $B = 119° \, 40'$, $c = 27° \, 47'$
3. $A = 58° \, 16'$, $B = 69° \, 20'$
4. $b = 77° \, 21.7'$, $B = 83° \, 56.7'$

103. Quadrantal Triangles. ***A quadrantal triangle*** is a spherical triangle one of whose sides equals 90°. Since each side of a spherical triangle is the supplement of the corresponding angle of its polar triangle, the polar triangle of a quadrantal triangle is clearly a right spherical triangle. Hence, by solving this polar triangle, and taking the supplements of its computed parts, we obtain the unknown parts of the original quadrantal triangle.

Example. Given a quadrantal triangle in which $c = 90°$, $C = 115° \, 20'$, $A = 42° \, 10'$, solve the triangle.

Denoting the polar triangle, § 99, by $A'B'C'$ and its sides by a', b', c', we have

$$C' = 90°, \quad a' = 137° \, 50', \quad c' = 64° \, 40'.$$

* From each logarithm, 10 is to be subtracted.

The solution of this right polar triangle gives:

$$B' = 115° \, 23.3',$$
$$A' = 132° \, 2.2',$$
$$b' = 125° \, 15.6'.$$

By taking the supplements of these values, we have the solution.

MISCELLANEOUS EXERCISES

Solve each of the following right spherical triangles.

1. $b = 40°$, $c = 63° \, 56'$
2. $a = 103° \, 40.2'$, $b = 62° \, 29.3'$
3. $a = 54° \, 30'$, $B = 44° \, 50'$
4. $B = 81° \, 24.4'$, $a = 34° \, 34.2'$
5. $a = 52° \, 0.8'$, $b = 25° \, 12.8'$
6. $A = 83° \, 56.7'$, $c = 78° \, 53.3'$
7. $c = 91° \, 42'$, $B = 95° \, 6'$
8. $b = 55° \, 28'$, $a = 63° \, 15'$
9. $A = 50° \, 0.8'$, $B = 79° \, 57.3'$
10. $A = 58° \, 16'$, $B = 69° \, 20'$
11. $A = 35° \, 29.6'$, $b = 75° \, 5.3'$
12. $b = 25° \, 47'$, $B = 35° \, 19'$
13. $A = 30° \, 57'$, $B = 69° \, 26'$
14. $a = 160° \, 12.2'$, $A = 150° \, 37'$
15. $A = 36° \, 44'$, $b = 30° \, 32.4'$
16. $a = 137° \, 50'$, $c = 64° \, 40'$

Solve each of the following quadrantal triangles, c being 90°.

17. $B = 65° \, 36'$, $a = 49° \, 30'$
18. $A = 83° \, 20.6'$, $B = 77° \, 14.3'$
19. $a = 90°$, $b = 90°$
20. $a = 100°$, $A = 100°$
21. $A = 72° \, 12.1'$, $B = 69° \, 13.8'$
22. $a = 112° \, 6.5'$, $C = 74° \, 30'$

23. An astronomer observes that the angular distance of the sun from the south point of the horizon is 75° and from the west point is 60°. Calculate the altitude of the sun above the horizon.

CHAPTER XIV

OBLIQUE SPHERICAL TRIANGLES

104. Law of Sines. *In any spherical triangle ABC,*

$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$

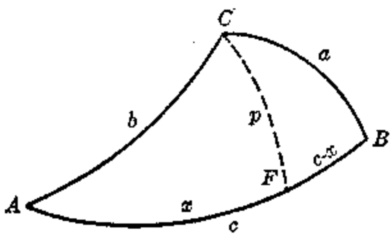


FIG. 122

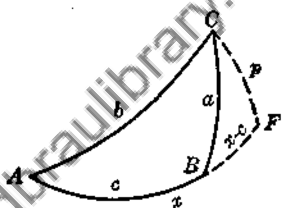


FIG. 123

PROOF. In the spherical triangle ABC (Fig. 122, or Fig. 123) draw the altitude $CF = p$, forming the right spherical triangles AFC and BFC. Applying formula (1), § 100, to each of these triangles, we have in either figure

$$\frac{\sin p}{\sin b} = \sin A, \qquad \frac{\sin p}{\sin a} = \sin B.$$

Dividing member by member, we obtain

(a) $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$

In a similar manner, we may write

(b) $$\frac{\sin b}{\sin c} = \frac{\sin B}{\sin C},$$

(c) $$\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}.$$

These results are usually written in the more symmetrical form

(1) $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

In this form, the law of sines for a spherical triangle can be stated as follows:

In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.

The law of sines may be used to compute a side when two angles and the side opposite one of them are given; or to compute an angle when two sides and the angle opposite one of them are known.

105. Formulas for cos a, cos b, cos c. *In any spherical triangle ABC,*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Proof. From the right spherical triangle BFC in Fig. 122, or Fig. 123, we have upon using (7), § 100, with $AF = x$,

(a) $$\cos a = \cos p \cos (c - x).$$

Similarly from the triangle AFC we have

(b) $$\cos b = \cos p \cos x.$$

Dividing (a) by (b), member by member, we find

(c) $$\frac{\cos a}{\cos b} = \frac{\cos (c - x)}{\cos x}.$$

From (8), § 51, and from (2), § 10, the formula (c) may be written in the form

(d) $$\frac{\cos a}{\cos b} = \cos c + \sin c \tan x.$$

But in the right spherical triangle AFC, we have from (2), § 100,

(e) $$\tan x = \tan b \cos A.$$

Replacing $\tan x$ in (d) by its value in (e) and multiplying both sides of the equation by $\cos b$, we have the relation

(2) $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Since a is any side of the spherical triangle ABC, we may write also

(3) $$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

(4) $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Formulas (2), (3), and (4) are known as the ***law of cosines*** for spherical triangles. They may be used to compute a side when two sides and their included angle are given; or to compute the three angles when the three sides are known. It is evident that these formulas are not well adapted to logarithmic computation.

106. Formulas for cos *A*, cos *B*, cos *C*. By considering the polar triangle $A'B'C'$ of the triangle ABC, the law of cosines may be made to assume a different form which is suitable for calculating an angle when two angles and their included side are given, or to compute the sides when the three angles are known.

Applying (2), § 105, to the polar triangle $A'B'C'$, we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Replacing a', b', c', and A' by $180° - A$, $180° - B$, $180° - C$, and $180° - a$ respectively (see § 99) and reducing, we obtain

$$\textbf{(5)} \qquad \cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

In a similar manner, we may write

$$(6) \qquad \cos B = -\cos C \cos A + \sin C \sin A \cos b,$$

$$(7) \qquad \cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

107. Half-Angle Formulas. Formulas convenient for computing by logarithms the angles of a spherical triangle when the three sides are given may be obtained from formulas (2), (3), and (4), § 105. Thus, from § 53, we recall that

$$\tan \frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}},$$

and from (2), § 105, we see that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Substituting for cos A, we obtain

$$\tan \frac{A}{2} = \sqrt{\frac{1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}{1 + \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}}$$

$$= \sqrt{\frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c + \cos a - \cos b \cos c}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{\cos(b-c) - \cos a}{\cos a - \cos(b+c)}} \qquad \text{(see §§ 48, 51)},$$

$$= \sqrt{\frac{\sin\frac{a+b-c}{2}\sin\frac{b-c-a}{2}}{\sin\frac{a+b+c}{2}\sin\frac{a-b-c}{2}}} \qquad \text{(see (24), § 55)}.$$

Finally,

$$(8) \qquad \tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},$$

where $2s = a + b + c$ (see § 47). In a similar manner, we have

$$(9) \qquad \tan\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}},$$

$$(10) \qquad \tan\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}.$$

By setting

$$k^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s},$$

we may write formulas (8), (9), and (10) in a form more convenient for logarithmic computation. They become

$$(11) \qquad \tan\frac{A}{2} = \frac{k}{\sin(s-a)},$$

$$(12) \qquad \tan\frac{B}{2} = \frac{k}{\sin(s-b)},$$

$$(13) \qquad \tan\frac{C}{2} = \frac{k}{\sin(s-c)}.$$

108. Half-Side Formulas. Formulas for computing by logarithms the sides of a spherical triangle when its angles are given may be obtained by applying formulas (11), (12), and (13), § 107, to the polar triangle $A'B'C'$ of ABC. Thus, for the polar triangle $A'B'C'$, formula (11), § 107, becomes

$$(a) \qquad \tan\frac{A'}{2} = \frac{k'}{\sin(s'-a')}$$

where $\qquad 2s' = a' + b' + c'$, and

$$k'^2 = \frac{\sin(s' - a')\sin(s' - b')\sin(s' - c')}{\sin s'}.$$

If we set $2S = A + B + C$, and recall from § 99 that

$$a' = 180° - A,\ b' = 180° - B,\ c' = 180° - C,\ A' = 180° - a,$$

we find that

$$s' = 270° - S, \qquad s' - b' = 90° - (S - B),$$
$$s' - a' = 90° - (S - A), \qquad s' - c' = 90° - (S - C).$$

Finally, if we set $K = 1/k'$ and make the appropriate substitutions and reductions in (a), we have

(14) $$\tan\frac{a}{2} = K\cos(S - A),$$

where

$$K = \frac{-\cos S}{\cos(S - A)\cos(S - B)\cos(S - C)}.$$

Similarly,

(15) $$\tan\frac{b}{2} = K\cos(S - B),$$

(16) $$\tan\frac{c}{2} = K\cos(S - C).$$

109. Napier's Analogies. In order to solve a spherical triangle by logarithms when we have given two sides and their included angle, or two angles and their included side, we make use of a set of formulas known as ***Napier's analogies.*** They may be derived as follows. From (11) and (12), § 107, we have by division of members,

(a) $$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \frac{\sin(s - b)}{\sin(s - a)}.$$

From a simple theorem of proportion, we may write (a) in the form

(b) $$\frac{\tan(A/2) - \tan(B/2)}{\tan(A/2) + \tan(B/2)} = \frac{\sin(s - b) - \sin(s - a)}{\sin(s - b) + \sin(s - a)}.$$

By simple reductions, using (2), § 10; (1), § 48; (7), § 51; (21) and (22), § 55; and the fact that $2s = a + b + c$, we may write (b) in the form

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}. \tag{17}$$

Similarly,

$$\tan\frac{b-c}{2} = \frac{\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}}\tan\frac{a}{2}, \tag{18}$$

$$\tan\frac{c-a}{2} = \frac{\sin\frac{C-A}{2}}{\sin\frac{C+A}{2}}\tan\frac{b}{2}. \tag{19}$$

By multiplication, formulas (11) and (12), § 107, yield the relation

$$\tan (A/2) \tan (B/2) = \sin (s - c)/\sin s,$$

which may be written in the form

$$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin (s-c)}{\sin s}. \tag{c}$$

By use of a simple theorem of proportion, (c) may be written in the form

$$\frac{\cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)}$$

and then may be reduced to the form

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}. \tag{20}$$

Similarly,

(21) $$\tan\frac{b+c}{2}=\frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}\tan\frac{a}{2},$$

(22) $$\tan\frac{c+a}{2}=\frac{\cos\frac{C-A}{2}}{\cos\frac{C+A}{2}}\tan\frac{b}{2}.$$

By applying formulas (17) to (22) in turn to the polar triangle $A'B'C'$ of the triangle ABC and by using the relations in § 99, we find upon reduction that

(23) $$\tan\frac{A-B}{2}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2},$$

(24) $$\tan\frac{B-C}{2}=\frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\cot\frac{A}{2},$$

(25) $$\tan\frac{C-A}{2}=\frac{\sin\frac{c-a}{2}}{\sin\frac{c+a}{2}}\cot\frac{B}{2},$$

(26) $$\tan\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2},$$

(27) $$\tan\frac{B+C}{2}=\frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\cot\frac{A}{2},$$

(28) $$\tan\frac{C+A}{2}=\frac{\cos\frac{c-a}{2}}{\cos\frac{c+a}{2}}\cot\frac{B}{2}.$$

110. Law of Species. Two angles (or two sides) of a spherical triangle are said to be of the same species if they are both acute or both obtuse; and of different species if one is acute and the other obtuse.

In the formula

$$\tan\frac{a+b}{2} = \frac{\cos\dfrac{A-B}{2}}{\cos\dfrac{A+B}{2}}\tan\frac{c}{2}, \quad \text{[see (20), § 109]}$$

$c/2$ and $(A - B)/2$ are each less than 90°, and $(a + b)/2$ and $(A + B)/2$ are each less than 180° (see § 98). It now follows that $\tan(c/2)$ and $\cos(A - B)/2$ are each positive, and that $\tan(a + b)/2$ and $\cos(A + B)/2$ are either both positive or both negative. That is to say, $(a + b)/2$ and $(A + B)/2$ are of the same species. Hence the law of species of a spherical triangle may be stated as follows:

One-half the sum of any two sides of a spherical triangle and one-half the sum of the opposite angles are of the same species.

111. Solution of Oblique Spherical Triangles. In general, if any three of the six parts of an oblique spherical triangle are given, the remaining parts may be found. The process of finding the unknown parts is called the ***solution*** of the triangle. Six cases arise, according as the given parts are:

CASE I. *Three sides.*
CASE II. *Three angles.*
CASE III. *Two angles and their included side.*
CASE IV. *Two sides and their included angle.*
CASE V. *Two sides and an angle opposite one of them.*
CASE VI. *Two angles and a side opposite one of them.*

A triangle, determined by the data in any one of the above cases, may be solved by means of logarithms, using the law of sines, tangents of half-angles or sides, and Napier's analogies. To facilitate the solution of triangles, the following table, which gives not only the formulas for solution but the formulas for checking, may be consulted. The table is constructed for a typical set of data under each case.

Case	Given	Solution	Check
I	a, b, c	(11), (12), (13) § 107	(1) § 104
II	A, B, C	(14), (15), (16) § 108	(1) § 104
III	A, B, c	(17), (20) § 109; (1) § 104	§ 110; (23) or (26) § 109
IV	a, b, C	(23), (26) § 109; (1) § 104	§ 110; (17) or (20) § 109
V	a, b, A	(1) § 104; § 110; (17) or (20), (23) or (26) § 109	(20) or (17), (26) or (23) § 109.
VI	A, B, a	(1) § 104; § 110; (17) or (20), (23) or (26) § 109.	(20) or (17), (26) or (23) § 109.

112. Comments on the Logarithmic Solution of Spherical Triangles. Since the logarithmic work necessary to solve a spherical triangle is similar to that employed for the solution of a plane triangle, it seems unnecessary to work out in detail any examples. The following comments, however, concerning each case will be of assistance in interpreting and analyzing the results.

In Case I, the work is carried out as in § 81. Since $A/2$, $B/2$, and $C/2$ are each less than 90°, it is unnecessary to use the law of species, § 110. The procedure in Case II is the same as in Case I.

In Case III, the work is carried out in the same manner as in § 79, except that we compute half the sum of two sides and half their difference and then find the sides. The remaining angle is then computed by use of the law of sines. Since the sine of an angle and the sine of its supplement are equal, it is necessary to use the law of species (§ 110) to determine which angle is to be retained. The procedure for Case IV is the same as for Case III.

If the given data in Case V are a, b, and A, we find from the law of sines that

$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

From this relation, we obtain two, one, or no values of the angle B according as log sin B is less than zero, equal to zero, or greater than zero. See Examples 1, 2, and 3, § 77. Each value of B so found, if it is to be retained, must obey the law of species, § 110. The solution may then be completed by the formulas indicated in the table. The procedure in Case VI is identical with that in Case V.

EXERCISES

Solve each of the following spherical triangles.

1. $a = 100°$, $b = 110°$, $c = 120°$.
2. $a = 97° 35'$, $b = 27° 8.4'$, $c = 119° 8.4'$.
3. $a = 112° 7'$, $b = 127° 39'$, $c = 71° 12'$.
4. $a = 76° 29.3'$, $b = 93° 18.6'$, $c = 122° 7.7'$.
5. $A = 150°$, $B = 131°$, $C = 115°$.
6. $A = 94° 40'$, $B = 86° 20'$, $C = 76° 30'$.
7. $A = 97° 10'$, $B = 74° 50'$, $C = 95° 40'$.
8. $A = 108° 45'$, $B = 140° 50'$, $C = 139° 25'$.
9. $A = 78°$, $B = 41°$, $c = 108°$.
10. $B = 70°$, $C = 131° 20'$, $a = 116°$.
11. $A = 116° 20'$, $C = 70° 7'$, $b = 82° 39.5'$.
12. $a = 63° 17'$, $b = 107° 23'$, $C = 65° 50'$.
13. $a = 100° 30'$, $c = 40° 20'$, $B = 46° 40'$.
14. $b = 132° 46.7'$, $c = 59° 50.1'$, $A = 56° 28.4'$.
15. $a = 58°$, $b = 137°$, $B = 131°$.
16. $b = 90° 36'$, $c = 39° 40'$, $B = 50° 52'$.
17. $a = 88° 12'$, $c = 141° 20.5'$, $C = 127° 4.2'$.
18. $A = 143°$, $B = 71°$, $b = 40°$.
19. $B = 36° 20'$, $C = 46° 30'$, $b = 42° 12'$.
20. $A = 57° 52'$, $C = 49° 50'$, $c = 162° 30'$.
21. By setting $\tan \theta = \cot b/\cos A$, show that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

reduces to

$$\cos a = \frac{\cos b}{\sin \theta} \sin (\theta + c).$$

22. Use the formulas in Exercise 21 to calculate by logarithms the side a of the spherical triangle in which $b = 63° 17'$, $c = 107° 23'$, and $A = 65° 50'$.

23. By using the polar triangle, show that the formula in Exercise 21 becomes

$$\cos A = \frac{\cos B \sin (C - \theta)}{\sin \theta} \quad \text{when} \quad \tan \theta = \frac{\cot B}{\cos a}.$$

24. Given $B = 84° 35'$, $C = 117° 13'$, $a = 64° 28'$, using logarithms calculate A by the formulas in Exercise 23.

25. A spherical triangle ABC lies on a sphere of radius 10 in. If $A = 100°$, $B = 60°$, and $C = 70°$, find the length of each side of the triangle in inches. Use Table II for reducing angles to radians.

26. A spherical triangle whose sides are 1300 ft., 1100 ft., and 1400 feet lies on a sphere of radius 1000 ft. Find its angles. HINT. Use (6), page 125.

113. Area of a Spherical Triangle. In elementary solid geometry it is shown that the areas of two spherical triangles lying on the same sphere have the same ratio as their spherical excesses. The ***spherical excess E*** of a spherical triangle is defined by the relation

$$E = A + B + C - 180°. \tag{29}$$

A spherical triangle is said to be ***trirectangular*** if each of its angles is 90°. The spherical excess of a trirectangular spherical triangle is 90° and its area is one-eighth of the area of the sphere upon which it lies, that is, $\pi R^2/2$ where R is the radius of the sphere. If S is the area of a spherical triangle of spherical excess E, we have by using the theorem above

$$S : \pi R^2/2 = E : 90°,$$

from which it follows that

$$S = \pi R^2 E/180°, \tag{30}$$

where E is measured in degrees. Formula (30) shows that the spherical excess E expressed in radians becomes the area of the spherical triangle if it lies on a sphere of radius 1.

If the sides of a spherical triangle are given, we may use a theorem, not derived here, known as ***L'Huilier's Theorem*** to calculate E, that is,

$$\tan^2 \frac{E}{4} = \tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}, \tag{31}$$

where $2s = a + b + c$.

EXERCISES

1. Given $A = 72° 32'$, $B = 58° 6'$, $C = 101° 9'$, find E in degrees; in radians.
2. Find the area of the triangle in Exercise 1 if the triangle lies on a sphere of radius 10 feet.
3. Given $A = 100°$, $B = 58°$, $C = 62°$, and $R = 3$, find the area S.
4. Given $A = 170°$, $B = 135°$, $C = 115°$, and $R = 10$, find the area S.
5. Given $a = 98°$, $b = 110°$, $c = 115°$, and $R = 40$, find the area S.
6. Given $a = b = c = 109° 28.2'$ and $R = 1$, find the area S.
7. Given $a = 58°$, $b = 137°$, $B = 131°$, and $R = 10$, find the area S.
8. Given $a = b = c = 60°$, and $R = 6$ in. Find the area S.
9. If $a = b = \pi/3$ and $c = \pi/2$, show that $E = \text{arc} \cos 7/9$.
10. If $a = b$, and $C = \pi/2$, show that $\tan E = \sin^2 a/(2 \cos a)$.

CHAPTER XV

APPLICATIONS

114. Introduction. Spherical trigonometry has many applications in navigation, astronomy, and geodesy. By considering the earth a sphere of radius 3958.6 miles, the shortest distance between two points upon its surface may be found as well as the bearing of one point from the other. The shortest distance, hereafter written distance, is measured along the shorter arc of the great circle joining the points.

The bearing of a point A from the point B is the angle (or its supplement) which the great circle through A and B makes with the meridian through B (see § 15).

115. Longitude and Latitude. In the plane, the position of a point is fixed by giving its distances from two fixed lines, the axes of reference. Similarly, the position of a point P on the earth's surface is fixed by giving its distances measured along the arcs of two great circles from two reference circles. The great circles commonly used as reference circles are the equator and the meridian through Greenwich, England. The student will recall that a meridian is a great circle passing through the north and south poles of the earth.

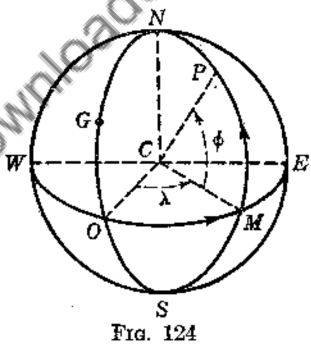


FIG. 124

In Fig. 124, the principal or zero meridian passing through Greenwich is marked $NGOS$ and the equator $EMOW$. The position of P is fixed if the arcs OM and MP are given. The values of these arcs, usually expressed in degrees, minutes, and tenths of a minute, are called the **longitude** λ and **latitude** ϕ respectively of the point P.

The direction in which the longitude is measured from the point O is indicated by attaching E or W to the number of degrees in the longitude which may range from 0° to 180°. Similarly, by attaching N or S to the number of degrees in the latitude we indicate that the point P lies in the northern or southern hemisphere respectively. For example, the longitude and latitude of Boston may be conveniently written as follows: Boston (long. 71° 5′ W, lat. 42° 21′ N).

116. The Terrestrial Triangle. Any two points A and B on the earth's surface whose longitudes and latitudes are known form with the north pole N a spherical triangle, called a ***terrestrial triangle.*** Since two sides and the included angle of the terrestrial triangle are known, its remaining parts may be found by the formulas under Case IV, § 111.

EXAMPLE. Find the distance in miles from Iowa City (long 91° 32′ W, lat. 41° 40′ N) to Liverpool (long. 3° 4′ W, lat. 53° 24′ N) and the bearing of Liverpool from Iowa City.

SOLUTION. In Fig. 125, let A, B, and N denote the positions of Iowa City, Liverpool, and the north pole respectively.

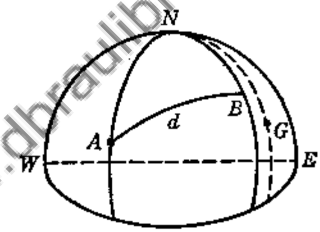


FIG. 125

$$\text{Arc } AN = 90° - 41° 40' = 48° 20',$$
$$\text{Arc } BN = 90° - 53° 24' = 36° 36',$$
$$\text{Angle } N = 91° 32' - 3° 4' = 88° 28'.$$

Using Napier's analogies, (23) and (26), and the law of sines, we find by logarithms that

$$A = 45° 20', \qquad B = 63°, \qquad d = 56° 56'.$$

Since 1′ of arc on a great circle of the earth is one *knot* or *nautical mile,* it follows that 1′ of arc = 6080 feet = 1.1515 miles. Hence to find an arc d of a great circle in miles, multiply the arc d expressed in minutes by 1.1515. We find

$$d = 3934 \text{ miles}.$$

The distance d may also be found by (3), § 85. Since $A = 45° 20'$, it follows that Liverpool bears N 45° 20′ E from Iowa City (see Fig. 125).

EXERCISES

1. Find the distance from San Francisco (long. 122° 26′ W, lat. 37° 47′ N) to New York (long. 73° 58′ W, lat. 40° 49′ N) and the bearing of New York from San Francisco.

2. Find the distance from New York (long. 73° 58′ W, lat. 40° 49′ N) to Greenwich (long. 0°, lat. 51° 29′ N).

3. Find the distance from Chicago (long. 87° 37′ W, lat. 41° 50′ N) to Montevideo (long. 56° 13′ W, lat. 34° 55′ S). HINT. Use the law of cosines.

4. Find the distance from San Francisco (long. 122° 26′ W, lat. 37° 47′ N) to Wellington, N. Z. (long. 174° 46′ E, lat. 41° 17′ S).

5. Find the shortest time for an airplane making an average speed of 150 mi. per hour, to fly from Quebec (long. 71° 13′ W, lat. 46° 48′ N) to Flagstaff (long. 111° 41′ W, lat. 35° 13′ N).

6. An airplane flies due west from Washington (long. 77° 4′ W, lat. 38° 55′ N) on the arc of a great circle. If a uniform speed of 175 miles per hour is maintained, find the longitude and latitude of the airplane after 10 hours of continuous flying.

117. Astronomical Data. Due to the daily rotation of the earth upon its axis, the heavenly bodies seem to rotate from east to west upon the surface of an immense sphere, called the ***celestial sphere.*** Only one-half of the celestial sphere is visible to a person on the earth's surface. The celestial sphere is represented in Fig. 126 with the earth, considered a point, as its center. In order to use spherical trigonometry to solve certain problems which arise in astronomy, we need to know the meaning of each of the following terms, all of which are represented in Fig. 126.

The ***horizon***, the great circle $N'E'S'W'$, formed by the intersection of the celestial sphere and the horizontal plane of the observer.

The ***zenith*** Z and the ***nadir*** Z' are the points of intersection of the celestial sphere and the vertical line passing through the observer.

The ***celestial north pole*** N and the ***celestial south pole*** S are the points of intersection of the earth's axis and the celestial sphere. The celestial sphere apparently rotates about the axis NS once in 24 hours.

The ***celestial equator***, the great circle $E'RW'W$, is the intersection of the celestial sphere and the plane containing the earth's equator.

The ***celestial meridians*** are great circles passing through the celestial poles N and S. The meridian NPM is called the ***hour circle*** of the point P.

118. The Astronomical Triangle. Let P be any point on the celestial sphere (Fig. 126). Draw its hour circle NPM and its zenith circle ZPQ. The arc QP is called the ***altitude*** of P. The arc MP is called the declination of P. The declination is considered positive when P lies north of the celestial equator and negative when P lies south of it. The arc WZ is called the ***latitude of the zenith*** and is equal to the latitude of the observer on the earth. The angle PNZ is called the *hour angle* of the point P.

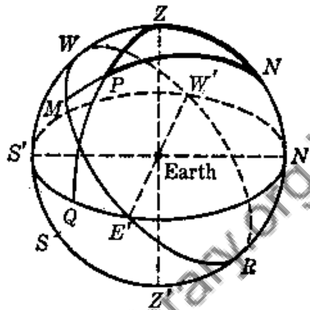


FIG. 126

Since the celestial sphere (apparently) rotates through 360° in 24 hours, one hour of time is equivalent to 15° of arc or hour angle. Consequently we may express an hour angle of a heavenly body in terms of hours, minutes, and tenths of a minute, and determine the time required for the body to rotate into the vertical or zenith meridian.

The points P, N, and Z determine a spherical triangle, called the ***astronomical triangle***. If sufficient data are given to determine any three of its parts, its remaining parts may be calculated (see § 111).

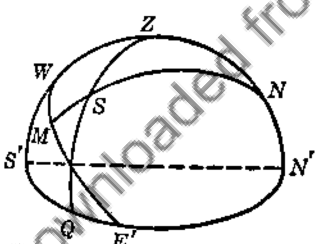


FIG. 127

EXAMPLE. In the forenoon, an observer at 49° 17′ North latitude finds the sun's altitude to be 30° and its declination 20°. Find the time of the observation.

SOLUTION. In Fig. 127, let S denote the position of the sun, N the celestial north pole, and Z the zenith of the observer. Then in the spherical triangle ZNS, we have

$$\text{Arc } NZ = 90° - 49° 17' = 40° 43',$$
$$\text{Arc } SZ = 90° - \text{arc } QS = 90° - 30° = 60°,$$
$$\text{Arc } SN = 90° - \text{arc } MS = 90° - 20° = 70°.$$

By logarithms, a short calculation, using (11), § 107, gives the hour angle N of the sun to be 66° 52.4′.

Dividing N by 15, we express N in hours, minutes, and tenths of a minute, that is,

$$N = 4 \text{ hours } 27.5 \text{ minutes.}$$

Therefore we conclude that the observation was made at 32.5 minutes after seven o'clock.

EXERCISES

1. In north latitude 38° 44′, the altitude of α andromedae whose declination is 28° 44′, was observed before midnight to be 55°. Find the time required until the star lies in the zenith meridian.

2. In the afternoon a civil engineer, whose latitude is 35° 20′ N, observed the altitude of the sun to be 20°. If the sun's declination was − 10°, find the time of the observation.

3. Find the time of sunset at Iowa City (lat. 41° 40′ N) when the sun's declination is − 20° 15′.

4. Find the altitude of a star at 3 A.M. in 40° North latitude if its declination is 17° 40′.

5. Find the time of sunrise at a place (lat. 12° 50′ N) on the date when the sun's declination is 15° 30′.

LOGARITHMIC AND TRIGONOMETRIC TABLES

TABLE I

COMMON LOGARITHMS OF NUMBERS

FROM

1 TO 10 000

TO

FIVE DECIMAL PLACES

N.	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25	24
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6

	23	22
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	21
1	2.1
2	4.2
3	6.3
4	8.4
5	10.5
6	12.6
7	14.7
8	16.8
9	18.9

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 2 =.30102 99957

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20	19
1	2.0	1.9
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

M
$= \log_{10} e$
$= \log_{10} 2.718 \cdots$
$= .43429\ 44819$

	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

$\log 3 = .47712\ 12547$

$\log \pi = .49714\ 98727$

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

$\log M = \log [\log e] = 9.63778\,431 - 10$

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

Prop. Pts.

450 — Logarithms of Numbers — 500

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
450	65 321	331	341	350	360	369	379	389	398	408	
451	418	427	437	447	456	466	475	485	495	504	
452	514	523	533	543	552	562	571	581	591	600	
453	610	619	629	639	648	658	667	677	686	696	
454	706	715	725	734	744	753	763	772	782	792	
455	801	811	820	830	839	849	858	868	877	887	
456	896	906	916	925	935	944	954	963	973	982	
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	
458	66 087	096	106	115	124	134	143	153	162	172	
459	181	191	200	210	219	229	238	247	257	266	
460	276	285	295	304	314	323	332	342	351	361	
461	370	380	389	398	408	417	427	436	445	455	
462	464	474	483	492	502	511	521	530	539	549	
463	558	567	577	586	596	605	614	624	633	642	
464	652	661	671	680	689	699	708	717	727	736	
465	745	755	764	773	783	792	801	811	820	829	
466	839	848	857	867	876	885	894	904	913	922	
467	932	941	950	960	969	978	987	997	*006	*015	
468	67 025	034	043	052	062	071	080	089	099	108	
469	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	
471	302	311	321	330	339	348	357	367	376	385	
472	394	403	413	422	431	440	449	459	468	477	
473	486	495	504	514	523	532	541	550	560	569	
474	578	587	596	605	614	624	633	642	651	660	
475	669	679	688	697	706	715	724	733	742	752	
476	761	770	779	788	797	806	815	825	834	843	
477	852	861	870	879	888	897	906	916	925	934	
478	943	952	961	970	979	988	997	*006	*015	*024	
479	68 034	043	052	061	070	079	088	097	106	115	
480	124	133	142	151	160	169	178	187	196	205	
481	215	224	233	242	251	260	269	278	287	296	
482	305	314	323	332	341	350	359	368	377	386	
483	395	404	413	422	431	440	449	458	467	476	
484	485	494	502	511	520	529	538	547	556	565	
485	574	583	592	601	610	619	628	637	646	655	
486	664	673	681	690	699	708	717	726	735	744	
487	753	762	771	780	789	797	806	815	824	833	
488	842	851	860	869	878	886	895	904	913	922	
489	931	940	949	958	966	975	984	993	*002	*011	
490	69 020	028	037	046	055	064	073	082	090	099	
491	108	117	126	135	144	152	161	170	179	188	
492	197	205	214	223	232	241	249	258	267	276	
493	285	294	302	311	320	329	338	346	355	364	
494	373	381	390	399	408	417	425	434	443	452	
495	461	469	478	487	496	504	513	522	531	539	
496	548	557	566	574	583	592	601	609	618	627	
497	636	644	653	662	671	679	688	697	705	714	
498	723	732	740	749	758	767	775	784	793	801	
499	810	819	827	836	845	854	862	871	880	888	
500	897	906	914	923	932	940	949	958	966	975	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

Prop. Pts.

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
500	69 897	906	914	923	932	940	949	958	966	975	
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
502	70 070	079	088	096	105	114	122	131	140	148	
503	157	165	174	183	191	200	209	217	226	234	
504	243	252	260	269	278	286	295	303	312	321	log 5
505	329	338	346	355	364	372	381	389	398	406	=.69897 00043
506	415	424	432	441	449	458	467	475	484	492	
507	501	509	518	526	535	544	552	561	569	578	
508	586	595	603	612	621	629	638	646	655	663	
509	672	680	689	697	706	714	723	731	740	749	
510	757	766	774	783	791	800	808	817	825	834	
511	842	851	859	868	876	885	893	902	910	919	
512	927	935	944	952	961	969	978	986	995	*003	
513	71 012	020	029	037	046	054	063	071	079	088	
514	096	105	113	122	130	139	147	155	164	172	
515	181	189	198	206	214	223	231	240	248	257	
516	265	273	282	290	299	307	315	324	332	341	
517	349	357	366	374	383	391	399	408	416	425	
518	433	441	450	458	466	475	483	492	500	508	
519	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
521	684	692	700	709	717	725	734	742	750	759	
522	767	775	784	792	800	809	817	825	834	842	
523	850	858	867	875	883	892	900	908	917	925	
524	933	941	950	958	966	975	983	991	999	*008	
525	72 016	024	032	041	049	057	066	074	082	090	
526	099	107	115	123	132	140	148	156	165	173	
527	181	189	198	206	214	222	230	239	247	255	
528	263	272	280	288	296	304	313	321	329	337	
529	346	354	362	370	378	387	395	403	411	419	
530	428	436	444	452	460	469	477	485	493	501	
531	509	518	526	534	542	550	558	567	575	583	
532	591	599	607	616	624	632	640	648	656	665	
533	673	681	689	697	705	713	722	730	738	746	
534	754	762	770	779	787	795	803	811	819	827	
535	835	843	852	860	868	876	884	892	900	908	
536	916	925	933	941	949	957	965	973	981	989	
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	
538	73 078	086	094	102	111	119	127	135	143	151	
539	159	167	175	183	191	199	207	215	223	231	
540	239	247	255	263	272	280	288	296	304	312	
541	320	328	336	344	352	360	368	376	384	392	
542	400	408	416	424	432	440	448	456	464	472	
543	480	488	496	504	512	520	528	536	544	552	
544	560	568	576	584	592	600	608	616	624	632	
545	640	648	656	664	672	679	687	695	703	711	
546	719	727	735	743	751	759	767	775	783	791	
547	799	807	815	823	830	838	846	854	862	870	
548	878	886	894	902	910	918	926	933	941	949	
549	957	965	973	981	989	997	*005	*013	*020	*028	
550	74 036	044	052	060	068	076	084	092	099	107	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

Prop. Pts.

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

N.	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

N.	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

log 7 = .84509 80400

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

750 — Logarithms of Numbers — 800

N.	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

Prop. Pts.

Logarithms of Important Constants

n = number	Value of n	$\log_{10} n$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Naperian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812263
$\pi \div 180$ = radians in 1°	0.01745329	8.24187737
$\pi \div 10800$ = radians in 1′	0.0002908882	6.46372612
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811133	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37 (exact legal value)	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g (average value)	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231 (exact legal value)	2.3636120
ft. lb. per sec. in 1 H. P.	550 (exact legal value)	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

Several Numbers Very Accurately

π =	3.14159	26535	89793	23846	26433	83280
e =	2.71828	18284	59045	23536	02874	71353
M =	0.43429	44819	03251	82765	11289	18917
$1 \div M$ =	2.30258	50929	94045	68401	79914	54684
$\log_{10} \pi$ =	0.49714	98726	94133	85435	12682	88291
$\log_{10} M$ =	9.63778	43113	00536	78912		

Certain Convenient Values for $n = 1$ to $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\log_{10} n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

TABLE II

THE VALUES OF TRIGONOMETRIC FUNCTIONS

FROM

0° TO 90° AT INTERVALS OF 10′

ALSO

THE CONVERSION OF DEGREES TO RADIANS

TO

FOUR DECIMAL PLACES

Radians	Degrees	Sin	Tan	Ctn *	Cos		
.0000	0° 00′	.0000	.0000	——	1.0000	90° 00′	1.5708
.0029	10	.0029	.0029	343.77	1.0000	50	1.5679
.0058	20	.0058	.0058	171.89	1.0000	40	1.5650
.0087	30	.0087	.0087	114.59	1.0000	30	1.5621
.0116	40	.0116	.0116	85.940	.9999	20	1.5592
.0145	50	.0145	.0145	68.750	.9999	10	1.5563
.0175	1° 00′	.0175	.0175	57.290	.9998	89° 00′	1.5533
.0204	10	.0204	.0204	49.104	.9998	50	1.5504
.0233	20	.0233	.0233	42.964	.9997	40	1.5475
.0262	30	.0262	.0262	38.188	.9997	30	1.5446
.0291	40	.0291	.0291	34.368	.9996	20	1.5417
.0320	50	.0320	.0320	31.242	.9995	10	1.5388
.0349	2° 00′	.0349	.0349	28.636	.9994	88° 00′	1.5359
.0378	10	.0378	.0378	26.432	.9993	50	1.5330
.0407	20	.0407	.0407	24.542	.9992	40	1.5301
.0436	30	.0436	.0437	22.904	.9990	30	1.5272
.0465	40	.0465	.0466	21.470	.9989	20	1.5243
.0495	50	.0494	.0495	20.206	.9988	10	1.5213
.0524	3° 00′	.0523	.0524	19.081	.9986	87° 00′	1.5184
.0553	10	.0552	.0553	18.075	.9985	50	1.5155
.0582	20	.0581	.0582	17.169	.9983	40	1.5126
.0611	30	.0610	.0612	16.350	.9981	30	1.5097
.0640	40	.0640	.0641	15.605	.9980	20	1.5068
.0669	50	.0669	.0670	14.924	.9978	10	1.5039
.0698	4° 00′	.0698	.0699	14.301	.9976	86° 00′	1.5010
.0727	10	.0727	.0729	13.727	.9974	50	1.4981
.0756	20	.0756	.0758	13.197	.9971	40	1.4952
.0785	30	.0785	.0787	12.706	.9969	30	1.4923
.0814	40	.0814	.0816	12.251	.9967	20	1.4893
.0844	50	.0843	.0846	11.826	.9964	10	1.4864
.0873	5° 00′	.0872	.0875	11.430	.9962	85° 00′	1.4835
.0902	10	.0901	.0904	11.059	.9959	50	1.4806
.0931	20	.0929	.0934	10.712	.9957	40	1.4777
.0960	30	.0958	.0963	10.385	.9954	30	1.4748
.0989	40	.0987	.0992	10.078	.9951	20	1.4719
.1018	50	.1016	.1022	9.7882	.9948	10	1.4690
.1047	6° 00′	.1045	.1051	9.5144	.9945	84° 00′	1.4661
.1076	10	.1074	.1080	9.2553	.9942	50	1.4632
.1105	20	.1103	.1110	9.0098	.9939	40	1.4603
.1134	30	.1132	.1139	8.7769	.9936	30	1.4573
.1164	40	.1161	.1169	8.5555	.9932	20	1.4544
.1193	50	.1190	.1198	8.3450	.9929	10	1.4515
.1222	7° 00′	.1219	.1228	8.1443	.9925	83° 00′	1.4486
.1251	10	.1248	.1257	7.9530	.9922	50	1.4457
.1280	20	.1276	.1287	7.7704	.9918	40	1.4428
.1309	30	.1305	.1317	7.5958	.9914	30	1.4399
.1338	40	.1334	.1346	7.4287	.9911	20	1.4370
.1367	50	.1363	.1376	7.2687	.9907	10	1.4341
.1396	8° 00′	.1392	.1405	7.1154	.9903	82° 00′	1.4312
.1425	10	.1421	.1435	6.9682	.9899	50	1.4283
.1454	20	.1449	.1465	6.8269	.9894	40	1.4254
.1484	30	.1478	.1495	6.6912	.9890	30	1.4224
.1513	40	.1507	.1524	6.5606	.9886	20	1.4195
.1542	50	.1536	.1554	6.4348	.9881	10	1.4166
.1571	9° 00′	.1564	.1584	6.3138	.9877	81° 00′	1.4137
		Cos	Ctn	Tan	Sin	Degrees	Radians

* In Tables II and III, ctn is used for cotangent.

Radians	Degrees	Sin	Tan	Ctn	Cos		
.1571	9° 00′	.1564	.1584	6.3138	.9877	81° 00′	1.4137
.1600	10	.1593	.1614	6.1970	.9872	50	1.4108
.1629	20	.1622	.1644	6.0844	.9868	40	1.4079
.1658	30	.1650	.1673	5.9758	.9863	30	1.4050
.1687	40	.1679	.1703	5.8708	.9858	20	1.4021
.1716	50	.1708	.1733	5.7694	.9853	10	1.3992
.1745	10° 00′	.1736	.1763	5.6713	.9848	80° 00′	1.3963
.1774	10	.1765	.1793	5.5764	.9843	50	1.3934
.1804	20	.1794	.1823	5.4845	.9838	40	1.3904
.1833	30	.1822	.1853	5.3955	.9833	30	1.3875
.1862	40	.1851	.1883	5.3093	.9827	20	1.3846
.1891	50	.1880	.1914	5.2257	.9822	10	1.3817
.1920	11° 00′	.1908	.1944	5.1446	.9816	79° 00′	1.3788
.1949	10	.1937	.1974	5.0658	.9811	50	1.3759
.1978	20	.1965	.2004	4.9894	.9805	40	1.3730
.2007	30	.1994	.2035	4.9152	.9799	30	1.3701
.2036	40	.2022	.2065	4.8430	.9793	20	1.3672
.2065	50	.2051	.2095	4.7729	.9787	10	1.3643
.2094	12° 00′	.2079	.2126	4.7046	.9781	78° 00′	1.3614
.2123	10	.2108	.2156	4.6382	.9775	50	1.3584
.2153	20	.2136	.2186	4.5736	.9769	40	1.3555
.2182	30	.2164	.2217	4.5107	.9763	30	1.3526
.2211	40	.2193	.2247	4.4494	.9757	20	1.3497
.2240	50	.2221	.2278	4.3897	.9750	10	1.3468
.2269	13° 00′	.2250	.2309	4.3315	.9744	77° 00′	1.3439
.2298	10	.2278	.2339	4.2747	.9737	50	1.3410
.2327	20	.2306	.2370	4.2193	.9730	40	1.3381
.2356	30	.2334	.2401	4.1653	.9724	30	1.3352
.2385	40	.2363	.2432	4.1126	.9717	20	1.3323
.2414	50	.2391	.2462	4.0611	.9710	10	1.3294
.2443	14° 00′	.2419	.2493	4.0108	.9703	76° 00′	1.3265
.2473	10	.2447	.2524	3.9617	.9696	50	1.3235
.2502	20	.2476	.2555	3.9136	.9689	40	1.3206
.2531	30	.2504	.2586	3.8667	.9681	30	1.3177
.2560	40	.2532	.2617	3.8208	.9674	20	1.3148
.2589	50	.2560	.2648	3.7760	.9667	10	1.3119
.2618	15° 00′	.2588	.2679	3.7321	.9659	75° 00′	1.3090
.2647	10	.2616	.2711	3.6891	.9652	50	1.3061
.2676	20	.2644	.2742	3.6470	.9644	40	1.3032
.2705	30	.2672	.2773	3.6059	.9636	30	1.3003
.2734	40	.2700	.2805	3.5656	.9628	20	1.2974
.2763	50	.2728	.2836	3.5261	.9621	10	1.2945
.2793	16° 00′	.2756	.2867	3.4874	.9613	74° 00′	1.2915
.2822	10	.2784	.2899	3.4495	.9605	50	1.2886
.2851	20	.2812	.2931	3.4124	.9596	40	1.2857
.2880	30	.2840	.2962	3.3759	.9588	30	1.2828
.2909	40	.2868	.2994	3.3402	.9580	20	1.2799
.2938	50	.2896	.3026	3.3052	.9572	10	1.2770
.2967	17° 00′	.2924	.3057	3.2709	.9563	73° 00′	1.2741
.2996	10	.2952	.3089	3.2371	.9555	50	1.2712
.3025	20	.2979	.3121	3.2041	.9546	40	1.2683
.3054	30	.3007	.3153	3.1716	.9537	30	1.2654
.3083	40	.3035	.3185	3.1397	.9528	20	1.2625
.3113	50	.3062	.3217	3.1084	.9520	10	1.2595
.3142	18° 00′	.3090	.3249	3.0777	.9511	72° 00′	1.2566
		Cos	Ctn	Tan	Sin	Degrees	Radians

Radians	Degrees	Sin	Tan	Ctn	Cos		
.3142	18° 00′	.3090	.3249	3.0777	.9511	72° 00′	1.2566
.3171	10	.3118	.3281	3.0475	.9502	50	1.2537
.3200	20	.3145	.3314	3.0178	.9492	40	1.2508
.3229	30	.3173	.3346	2.9887	.9483	30	1.2479
.3258	40	.3201	.3378	2.9600	.9474	20	1.2450
.3287	50	.3228	.3411	2.9319	.9465	10	1.2421
.3316	19° 00′	.3256	.3443	2.9042	.9455	71° 00′	1.2392
.3345	10	.3283	.3476	2.8770	.9446	50	1.2363
.3374	20	.3311	.3508	2.8502	.9436	40	1.2334
.3403	30	.3338	.3541	2.8239	.9426	30	1.2305
.3432	40	.3365	.3574	2.7980	.9417	20	1.2275
.3462	50	.3393	.3607	2.7725	.9407	10	1.2246
.3491	20° 00′	.3420	.3640	2.7475	.9397	70° 00′	1.2217
.3520	10	.3448	.3673	2.7228	.9387	50	1.2188
.3549	20	.3475	.3706	2.6985	.9377	40	1.2159
.3578	30	.3502	.3739	2.6746	.9367	30	1.2130
.3607	40	.3529	.3772	2.6511	.9356	20	1.2101
.3636	50	.3557	.3805	2.6279	.9346	10	1.2072
.3665	21° 00′	.3584	.3839	2.6051	.9336	69° 00′	1.2043
.3694	10	.3611	.3872	2.5826	.9325	50	1.2014
.3723	20	.3638	.3906	2.5605	.9315	40	1.1985
.3752	30	.3665	.3939	2.5386	.9304	30	1.1956
.3782	40	.3692	.3973	2.5172	.9293	20	1.1926
.3811	50	.3719	.4006	2.4960	.9283	10	1.1897
.3840	22° 00′	.3746	.4040	2.4751	.9272	68° 00′	1.1868
.3869	10	.3773	.4074	2.4545	.9261	50	1.1839
.3898	20	.3800	.4108	2.4342	.9250	40	1.1810
.3927	30	.3827	.4142	2.4142	.9239	30	1.1781
.3956	40	.3854	.4176	2.3945	.9228	20	1.1752
.3985	50	.3881	.4210	2.3750	.9216	10	1.1723
.4014	23° 00′	.3907	.4245	2.3559	.9205	67° 00′	1.1694
.4043	10	.3934	.4279	2.3369	.9194	50	1.1665
.4072	20	.3961	.4314	2.3183	.9182	40	1.1636
.4102	30	.3987	.4348	2.2998	.9171	30	1.1606
.4131	40	.4014	.4383	2.2817	.9159	20	1.1577
.4160	50	.4041	.4417	2.2637	.9147	10	1.1548
.4189	24° 00′	.4067	.4452	2.2460	.9135	66° 00′	1.1519
.4218	10	.4094	.4487	2.2286	.9124	50	1.1490
.4247	20	.4120	.4522	2.2113	.9112	40	1.1461
.4276	30	.4147	.4557	2.1943	.9100	30	1.1432
.4305	40	.4173	.4592	2.1775	.9088	20	1.1403
.4334	50	.4200	.4628	2.1609	.9075	10	1.1374
.4363	25° 00′	.4226	.4663	2.1445	.9063	65° 00′	1.1345
.4392	10	.4253	.4699	2.1283	.9051	50	1.1316
.4422	20	.4279	.4734	2.1123	.9038	40	1.1286
.4451	30	.4305	.4770	2.0965	.9026	30	1.1257
.4480	40	.4331	.4806	2.0809	.9013	20	1.1228
.4509	50	.4358	.4841	2.0655	.9001	10	1.1199
.4538	26° 00′	.4384	.4877	2.0503	.8988	64° 00′	1.1170
.4567	10	.4410	.4913	2.0353	.8975	50	1.1141
.4596	20	.4436	.4950	2.0204	.8962	40	1.1112
.4625	30	.4462	.4986	2.0057	.8949	30	1.1083
.4654	40	.4488	.5022	1.9912	.8936	20	1.1054
.4683	50	.4514	.5059	1.9768	.8923	10	1.1025
.4712	27° 00′	.4540	.5095	1.9626	.8910	63° 00′	1.0996
		Cos	Ctn	Tan	Sin	Degrees	Radians

Four Place Trigonometric Functions

Radians	Degrees	Sin	Tan	Ctn	Cos		
.4712	27° 00′	.4540	.5095	1.9626	.8910	63° 00′	1.0996
.4741	10	.4566	.5132	1.9486	.8897	50	1.0966
.4771	20	.4592	.5169	1.9347	.8884	40	1.0937
.4800	30	.4617	.5206	1.9210	.8870	30	1.0908
.4829	40	.4643	.5243	1.9074	.8857	20	1.0879
.4858	50	.4669	.5280	1.8940	.8843	10	1.0850
.4887	28° 00′	.4695	.5317	1.8807	.8829	62° 00′	1.0821
.4916	10	.4720	.5354	1.8676	.8816	50	1.0792
.4945	20	.4746	.5392	1.8546	.8802	40	1.0763
.4974	30	.4772	.5430	1.8418	.8788	30	1.0734
.5003	40	.4797	.5467	1.8291	.8774	20	1.0705
.5032	50	.4823	.5505	1.8165	.8760	10	1.0676
.5061	29° 00′	.4848	.5543	1.8040	.8746	61° 00′	1.0647
.5091	10	.4874	.5581	1.7917	.8732	50	1.0617
.5120	20	.4899	.5619	1.7796	.8718	40	1.0588
.5149	30	.4924	.5658	1.7675	.8704	30	1.0559
.5178	40	.4950	.5696	1.7556	.8689	20	1.0530
.5207	50	.4975	.5735	1.7437	.8675	10	1.0501
.5236	30° 00′	.5000	.5774	1.7321	.8660	60° 00′	1.0472
.5265	10	.5025	.5812	1.7205	.8646	50	1.0443
.5294	20	.5050	.5851	1.7090	.8631	40	1.0414
.5323	30	.5075	.5890	1.6977	.8616	30	1.0385
.5352	40	.5100	.5930	1.6864	.8601	20	1.0356
.5381	50	.5125	.5969	1.6753	.8587	10	1.0327
.5411	31° 00′	.5150	.6009	1.6643	.8572	59° 00′	1.0297
.5440	10	.5175	.6048	1.6534	.8557	50	1.0268
.5469	20	.5200	.6088	1.6426	.8542	40	1.0239
.5498	30	.5225	.6128	1.6319	.8526	30	1.0210
.5527	40	.5250	.6168	1.6212	.8511	20	1.0181
.5556	50	.5275	.6208	1.6107	.8496	10	1.0152
.5585	32° 00′	.5299	.6249	1.6003	.8480	58° 00′	1.0123
.5614	10	.5324	.6289	1.5900	.8465	50	1.0094
.5643	20	.5348	.6330	1.5798	.8450	40	1.0065
.5672	30	.5373	.6371	1.5697	.8434	30	1.0036
.5701	40	.5398	.6412	1.5597	.8418	20	1.0007
.5730	50	.5422	.6453	1.5497	.8403	10	.9977
.5760	33° 00′	.5446	.6494	1.5399	.8387	57° 00′	.9948
.5789	10	.5471	.6536	1.5301	.8371	50	.9919
.5818	20	.5495	.6577	1.5204	.8355	40	.9890
.5847	30	.5519	.6619	1.5108	.8339	30	.9861
.5876	40	.5544	.6661	1.5013	.8323	20	.9832
.5905	50	.5568	.6703	1.4919	.8307	10	.9803
.5934	34° 00′	.5592	.6745	1.4826	.8290	56° 00′	.9774
.5963	10	.5616	.6787	1.4733	.8274	50	.9745
.5992	20	.5640	.6830	1.4641	.8258	40	.9716
.6021	30	.5664	.6873	1.4550	.8241	30	.9687
.6050	40	.5688	.6916	1.4460	.8225	20	.9657
.6080	50	.5712	.6959	1.4370	.8208	10	.9628
.6109	35° 00′	.5736	.7002	1.4281	.8192	55° 00′	.9599
.6138	10	.5760	.7046	1.4193	.8175	50	.9570
.6167	20	.5783	.7089	1.4106	.8158	40	.9541
.6196	30	.5807	.7133	1.4019	.8141	30	.9512
.6225	40	.5831	.7177	1.3934	.8124	20	.9483
.6254	50	.5854	.7221	1.3848	.8107	10	.9454
.6283	36° 00′	.5878	.7265	1.3764	.8090	54° 00′	.9425
		Cos	Ctn	Tan	Sin	Degrees	Radians

Radians	Degrees	Sin	Tan	Ctn	Cos		
.6283	**36° 00′**	.5878	.7265	1.3764	.8090	**54° 00′**	.9425
.6312	10	.5901	.7310	1.3680	.8073	50	.9396
.6341	20	.5925	.7355	1.3597	.8056	40	.9367
.6370	30	.5948	.7400	1.3514	.8039	30	.9338
.6400	40	.5972	.7445	1.3432	.8021	20	.9308
.6429	50	.5995	.7490	1.3351	.8004	10	.9279
.6458	**37° 00′**	.6018	.7536	1.3270	.7986	**53° 00′**	.9250
.6487	10	.6041	.7581	1.3190	.7969	50	.9221
.6516	20	.6065	.7627	1.3111	.7951	40	.9192
.6545	30	.6088	.7673	1.3032	.7934	30	.9163
.6574	40	.6111	.7720	1.2954	.7916	20	.9134
.6603	50	.6134	.7766	1.2876	.7898	10	.9105
.6632	**38° 00′**	.6157	.7813	1.2799	.7880	**52° 00′**	.9076
.6661	10	.6180	.7860	1.2723	.7862	50	.9047
.6690	20	.6202	.7907	1.2647	.7844	40	.9018
.6720	30	.6225	.7954	1.2572	.7826	30	.8988
.6749	40	.6248	.8002	1.2497	.7808	20	.8959
.6778	50	.6271	.8050	1.2423	.7790	10	.8930
.6807	**39° 00′**	.6293	.8098	1.2349	.7771	**51° 00′**	.8901
.6836	10	.6316	.8146	1.2276	.7753	50	.8872
.6865	20	.6338	.8195	1.2203	.7735	40	.8843
.6894	30	.6361	.8243	1.2131	.7716	30	.8814
.6923	40	.6383	.8292	1.2059	.7698	20	.8785
.6952	50	.6406	.8342	1.1988	.7679	10	.8756
.6981	**40° 00′**	.6428	.8391	1.1918	.7660	**50° 00′**	.8727
.7010	10	.6450	.8441	1.1847	.7642	50	.8698
.7039	20	.6472	.8491	1.1778	.7623	40	.8668
.7069	30	.6494	.8541	1.1708	.7604	30	.8639
.7098	40	.6517	.8591	1.1640	.7585	20	.8610
.7127	50	.6539	.8642	1.1571	.7566	10	.8581
.7156	**41° 00′**	.6561	.8693	1.1504	.7547	**49° 00′**	.8552
.7185	10	.6583	.8744	1.1436	.7528	50	.8523
.7214	20	.6604	.8796	1.1369	.7509	40	.8494
.7243	30	.6626	.8847	1.1303	.7490	30	.8465
.7272	40	.6648	.8899	1.1237	.7470	20	.8436
.7301	50	.6670	.8952	1.1171	.7451	10	.8407
.7330	**42° 00′**	.6691	.9004	1.1106	.7431	**48° 00′**	.8378
.7359	10	.6713	.9057	1.1041	.7412	50	.8348
.7389	20	.6734	.9110	1.0977	.7392	40	.8319
.7418	30	.6756	.9163	1.0913	.7373	30	.8290
.7447	40	.6777	.9217	1.0850	.7353	20	.8261
.7476	50	.6799	.9271	1.0786	.7333	10	.8232
.7505	**43° 00′**	.6820	.9325	1.0724	.7314	**47° 00′**	.8203
.7534	10	.6841	.9380	1.0661	.7294	50	.8174
.7563	20	.6862	.9435	1.0599	.7274	40	.8145
.7592	30	.6884	.9490	1.0538	.7254	30	.8116
.7621	40	.6905	.9545	1.0477	.7234	20	.8087
.7650	50	.6926	.9601	1.0416	.7214	10	.8058
.7679	**44° 00′**	.6947	.9657	1.0355	.7193	**46° 00′**	.8029
.7709	10	.6967	.9713	1.0295	.7173	50	.7999
.7738	20	.6988	.9770	1.0235	.7153	40	.7970
.7767	30	.7009	.9827	1.0176	.7133	30	.7941
.7796	40	.7030	.9884	1.0117	.7112	20	.7912
.7825	50	.7050	.9942	1.0058	.7092	10	.7883
.7854	**45° 00′**	.7071	1.0000	1.0000	.7071	**45° 00′**	.7854
		Cos	Ctn	Tan	Sin	Degrees	Radians

TABLE III

COMMON LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

FROM

0° TO 90° AT INTERVALS OF ONE MINUTE

TO

FIVE DECIMAL PLACES

From each logarithm given, subtract 10

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	—		—		—	10.00 000	60
1	6.46 373	30103	6.46 373	30103	13.53 627	10.00 000	59
2	6.76 476	17609	6.76 476	17609	13.23 524	10.00 000	58
3	6.94 085	12494	6.94 085	12494	13.05 915	10.00 000	57
4	7.06 579	9691	7.06 579	9691	12.93 421	10.00 000	56
5	7.16 270	7918	7.16 270	7918	12.83 730	10.00 000	55
6	7.24 188	6694	7.24 188	6694	12.75 812	10.00 000	54
7	7.30 882	5800	7.30 882	5800	12.69 118	10.00 000	53
8	7.36 682	5115	7.36 682	5115	12.63 318	10.00 000	52
9	7.41 797	4576	7.41 797	4576	12.58 203	10.00 000	51
10	7.46 373	4139	7.46 373	4139	12.53 627	10.00 000	50
11	7.50 512	3779	7.50 512	3779	12.49 488	10.00 000	49
12	7.54 291	3476	7.54 291	3476	12.45 709	10.00 000	48
13	7.57 767	3218	7.57 767	3219	12.42 233	10.00 000	47
14	7.60 985	2997	7.60 986	2996	12.39 014	10.00 000	46
15	7.63 982	2802	7.63 982	2803	12.36 018	10.00 000	45
16	7.66 784	2633	7.66 785	2633	12.33 215	10.00 000	44
17	7.69 417	2483	7.69 418	2482	12.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	12.28 100	9.99 999	42
19	7.74 248	2227	7.74 248	2228	12.25 752	9.99 999	41
20	7.76 475	2119	7.76 476	2119	12.23 524	9.99 999	40
21	7.78 594	2021	7.78 595	2020	12.21 405	9.99 999	39
22	7.80 615	1930	7.80 615	1931	12.19 385	9.99 999	38
23	7.82 545	1848	7.82 546	1848	12.17 454	9.99 999	37
24	7.84 393	1773	7.84 394	1773	12.15 606	9.99 999	36
25	7.86 166	1704	7.86 167	1704	12.13 833	9.99 999	35
26	7.87 870	1639	7.87 871	1639	12.12 129	9.99 999	34
27	7.89 509	1579	7.89 510	1579	12.10 490	9.99 999	33
28	7.91 088	1524	7.91 089	1524	12.08 911	9.99 999	32
29	7.92 612	1472	7.92 613	1473	12.07 387	9.99 998	31
30	7.94 084	1424	7.94 086	1424	12.05 914	9.99 998	30
31	7.95 508	1379	7.95 510	1379	12.04 490	9.99 998	29
32	7.96 887	1336	7.96 889	1336	12.03 111	9.99 998	28
33	7.98 223	1297	7.98 225	1297	12.01 775	9.99 998	27
34	7.99 520	1259	7.99 522	1259	12.00 478	9.99 998	26
35	8.00 779	1223	8.00 781	1223	11.99 219	9.99 998	25
36	8.02 002	1190	8.02 004	1190	11.97 996	9.99 998	24
37	8.03 192	1158	8.03 194	1159	11.96 806	9.99 997	23
38	8.04 350	1128	8.04 353	1128	11.95 647	9.99 997	22
39	8.05 478	1100	8.05 481	1100	11.94 519	9.99 997	21
40	8.06 578	1072	8.06 581	1072	11.93 419	9.99 997	20
41	8.07 650	1046	8.07 653	1047	11.92 347	9.99 997	19
42	8.08 696	1022	8.08 700	1022	11.91 300	9.99 997	18
43	8.09 718	999	8.09 722	998	11.90 278	9.99 997	17
44	8.10 717	976	8.10 720	976	11.89 280	9.99 996	16
45	8.11 693	954	8.11 696	955	11.88 304	9.99 996	15
46	8.12 647	934	8.12 651	934	11.87 349	9.99 996	14
47	8.13 581	914	8.13 585	915	11.86 415	9.99 996	13
48	8.14 495	896	8.14 500	895	11.85 500	9.99 996	12
49	8.15 391	877	8.15 395	878	11.84 605	9.99 996	11
50	8.16 268	860	8.16 273	860	11.83 727	9.99 995	10
51	8.17 128	843	8.17 133	843	11.82 867	9.99 995	9
52	8.17 971	827	8.17 976	828	11.82 024	9.99 995	8
53	8.18 798	812	8.18 804	812	11.81 196	9.99 995	7
54	8.19 610	797	8.19 616	797	11.80 384	9.99 995	6
55	8.20 407	782	8.20 413	782	11.79 587	9.99 994	5
56	8.21 189	769	8.21 195	769	11.78 805	9.99 994	4
57	8.21 958	755	8.21 964	756	11.78 036	9.99 994	3
58	8.22 713	743	8.22 720	742	11.77 280	9.99 994	2
59	8.23 456	730	8.23 462	730	11.76 538	9.99 994	1
60	8.24 186		8.24 192		11.75 808	9.99 993	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

89° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	′
0	8.24 186	717	8.24 192	718	11.75 808	9.99 993	60
1	8.24 903	706	8.24 910	706	11.75 090	9.99 993	59
2	8.25 609	695	8.25 616	696	11.74 384	9.99 993	58
3	8.26 304	684	8.26 312	684	11.73 688	9.99 993	57
4	8.26 988	673	8.26 996	673	11.73 004	9.99 992	56
5	8.27 661	663	8.27 669	663	11.72 331	9.99 992	55
6	8.28 324	653	8.28 332	654	11.71 668	9.99 992	54
7	8.28 977	644	8.28 986	643	11.71 014	9.99 992	53
8	8.29 621	634	8.29 629	634	11.70 371	9.99 992	52
9	8.30 255	624	8.30 263	625	11.69 737	9.99 991	51
10	8.30 879	616	8.30 888	617	11.69 112	9.99 991	50
11	8.31 495	608	8.31 505	607	11.68 495	9.99 991	49
12	8.32 103	599	8.32 112	599	11.67 888	9.99 990	48
13	8.32 702	590	8.32 711	591	11.67 289	9.99 990	47
14	8.33 292	583	8.33 302	584	11.66 698	9.99 990	46
15	8.33 875	575	8.33 886	575	11.66 114	9.99 990	45
16	8.34 450	568	8.34 461	568	11.65 539	9.99 989	44
17	8.35 018	560	8.35 029	561	11.64 971	9.99 989	43
18	8.35 578	553	8.35 590	553	11.64 410	9.99 989	42
19	8.36 131	547	8.36 143	546	11.63 857	9.99 989	41
20	8.36 678	539	8.36 689	540	11.63 311	9.99 988	40
21	8.37 217	533	8.37 229	533	11.62 771	9.99 988	39
22	8.37 750	526	8.37 762	527	11.62 238	9.99 988	38
23	8.38 276	520	8.38 289	520	11.61 711	9.99 987	37
24	8.38 796	514	8.38 809	514	11.61 191	9.99 987	36
25	8.39 310	508	8.39 323	509	11.60 677	9.99 987	35
26	8.39 818	502	8.39 832	502	11.60 168	9.99 986	34
27	8.40 320	496	8.40 334	496	11.59 666	9.99 986	33
28	8.40 816	491	8.40 830	491	11.59 170	9.99 986	32
29	8.41 307	485	8.41 321	486	11.58 679	9.99 985	31
30	8.41 792	480	8.41 807	480	11.58 193	9.99 985	30
31	8.42 272	474	8.42 287	475	11.57 713	9.99 985	29
32	8.42 746	470	8.42 762	470	11.57 238	9.99 984	28
33	8.43 216	464	8.43 232	464	11.56 768	9.99 984	27
34	8.43 680	459	8.43 696	460	11.56 304	9.99 984	26
35	8.44 139	455	8.44 156	455	11.55 844	9.99 983	25
36	8.44 594	450	8.44 611	450	11.55 389	9.99 983	24
37	8.45 044	445	8.45 061	446	11.54 939	9.99 983	23
38	8.45 489	441	8.45 507	441	11.54 493	9.99 982	22
39	8.45 930	436	8.45 948	437	11.54 052	9.99 982	21
40	8.46 366	433	8.46 385	432	11.53 615	9.99 982	20
41	8.46 799	427	8.46 817	428	11.53 183	9.99 981	19
42	8.47 226	424	8.47 245	424	11.52 755	9.99 981	18
43	8.47 650	419	8.47 669	420	11.52 331	9.99 981	17
44	8.48 069	416	8.48 089	416	11.51 911	9.99 980	16
45	8.48 485	411	8.48 505	412	11.51 495	9.99 980	15
46	8.48 896	408	8.48 917	408	11.51 083	9.99 979	14
47	8.49 304	404	8.49 325	404	11.50 675	9.99 979	13
48	8.49 708	400	8.49 729	401	11.50 271	9.99 979	12
49	8.50 108	396	8.50 130	397	11.49 870	9.99 978	11
50	8.50 504	393	8.50 527	393	11.49 473	9.99 978	10
51	8.50 897	390	8.50 920	390	11.49 080	9.99 977	9
52	8.51 287	386	8.51 310	386	11.48 690	9.99 977	8
53	8.51 673	382	8.51 696	383	11.48 304	9.99 977	7
54	8.52 055	379	8.52 079	380	11.47 921	9.99 976	6
55	8.52 434	376	8.52 459	376	11.47 541	9.99 976	5
56	8.52 810	373	8.52 835	373	11.47 165	9.99 975	4
57	8.53 183	369	8.53 208	370	11.46 792	9.99 975	3
58	8.53 552	367	8.53 578	367	11.46 422	9.99 974	2
59	8.53 919	363	8.53 945	363	11.46 055	9.99 974	1
60	8.54 282		8.54 308		11.45 692	9.99 974	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	710	690	670	650
2	142	138	134	130
3	213	207	201	195
4	284	276	268	260
5	355	345	335	325
6	426	414	402	390
7	497	483	469	455
8	568	552	536	520
9	639	621	603	585
	630	620	610	600
2	126	124	122	120
3	189	186	183	180
4	252	248	244	240
5	315	310	305	300
6	378	372	366	360
7	441	434	427	420
8	504	496	488	480
9	567	558	549	540
	590	580	570	560
2	118	116	114	112
3	177	174	171	168
4	236	232	228	224
5	295	290	285	280
6	354	348	342	336
7	413	406	399	392
8	472	464	456	448
9	531	522	513	504
	550	540	530	520
2	110	108	106	104
3	165	162	159	156
4	220	216	212	208
5	275	270	265	260
6	330	324	318	312
7	385	378	371	364
8	440	432	424	416
9	495	486	477	468
	510	500	490	480
2	102	100	98	96
3	153	150	147	144
4	204	200	196	192
5	255	250	245	240
6	306	300	294	288
7	357	350	343	336
8	408	400	392	384
9	459	450	441	432
	470	460	450	440
2	94	92	90	88
3	141	138	135	132
4	188	184	180	176
5	235	230	225	220
6	282	276	270	264
7	329	322	315	308
8	376	368	360	352
9	423	414	405	396
	430	420	410	400
2	86	84	82	80
3	129	126	123	120
4	172	168	164	160
5	215	210	205	200
6	258	252	246	240
7	301	294	287	280
8	344	336	328	320
9	387	378	369	360
	390	380	370	360
2	78	76	74	72
3	117	114	111	108
4	156	152	148	144
5	195	190	185	180
6	234	228	222	216
7	273	266	259	252
8	312	304	296	288
9	351	342	333	324

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	8.54 282	360	8.54 308	361	11.45 692	9.99 974	60
1	8.54 642	357	8.54 669	358	11.45 331	9.99 973	59
2	8.54 999	355	8.55 027	355	11.44 973	9.99 973	58
3	8.55 354	351	8.55 382	352	11.44 618	9.99 972	57
4	8.55 705	349	8.55 734	349	11.44 266	9.99 972	56
5	8.56 054	346	8.56 083	346	11.43 917	9.99 971	55
6	8.56 400	343	8.56 429	344	11.43 571	9.99 971	54
7	8.56 743	341	8.56 773	341	11.43 227	9.99 970	53
8	8.57 084	337	8.57 114	338	11.42 886	9.99 970	52
9	8.57 421	336	8.57 452	336	11.42 548	9.99 969	51
10	8.57 757	332	8.57 788	333	11.42 212	9.99 969	50
11	8.58 089	330	8.58 121	330	11.41 879	9.99 968	49
12	8.58 419	328	8.58 451	328	11.41 549	9.99 968	48
13	8.58 747	325	8.58 779	326	11.41 221	9.99 967	47
14	8.59 072	323	8.59 105	323	11.40 895	9.99 967	46
15	8.59 395	320	8.59 428	321	11.40 572	9.99 967	45
16	8.59 715	318	8.59 749	319	11.40 251	9.99 966	44
17	8.60 033	316	8.60 068	316	11.39 932	9.99 966	43
18	8.60 349	313	8.60 384	314	11.39 616	9.99 965	42
19	8.60 662	311	8.60 698	311	11.39 302	9.99 964	41
20	8.60 973	309	8.61 009	310	11.38 991	9.99 964	40
21	8.61 282	307	8.61 319	307	11.38 681	9.99 963	39
22	8.61 589	305	8.61 626	305	11.38 374	9.99 963	38
23	8.61 894	302	8.61 931	303	11.38 069	9.99 962	37
24	8.62 196	301	8.62 234	301	11.37 766	9.99 962	36
25	8.62 497	298	8.62 535	299	11.37 465	9.99 961	35
26	8.62 795	296	8.62 834	297	11.37 166	9.99 961	34
27	8.63 091	294	8.63 131	295	11.36 869	9.99 960	33
28	8.63 385	293	8.63 426	292	11.36 574	9.99 960	32
29	8.63 678	290	8.63 718	291	11.36 282	9.99 959	31
30	8.63 968	288	8.64 009	289	11.35 991	9.99 959	30
31	8.64 256	287	8.64 298	287	11.35 702	9.99 958	29
32	8.64 543	284	8.64 585	285	11.35 415	9.99 958	28
33	8.64 827	283	8.64 870	284	11.35 130	9.99 957	27
34	8.65 110	281	8.65 154	281	11.34 846	9.99 956	26
35	8.65 391	279	8.65 435	280	11.34 565	9.99 956	25
36	8.65 670	277	8.65 715	278	11.34 285	9.99 955	24
37	8.65 947	276	8.65 993	276	11.34 007	9.99 955	23
38	8.66 223	274	8.66 269	274	11.33 731	9.99 954	22
39	8.66 497	272	8.66 543	273	11.33 457	9.99 954	21
40	8.66 769	270	8.66 816	271	11.33 184	9.99 953	20
41	8.67 039	269	8.67 087	269	11.32 913	9.99 952	19
42	8.67 308	267	8.67 356	268	11.32 644	9.99 952	18
43	8.67 575	266	8.67 624	266	11.32 376	9.99 951	17
44	8.67 841	263	8.67 890	264	11.32 110	9.99 951	16
45	8.68 104	263	8.68 154	263	11.31 846	9.99 950	15
46	8.68 367	260	8.68 417	261	11.31 583	9.99 949	14
47	8.68 627	259	8.68 678	260	11.31 322	9.99 949	13
48	8.68 886	258	8.68 938	258	11.31 062	9.99 948	12
49	8.69 144	256	8.69 196	257	11.30 804	9.99 948	11
50	8.69 400	254	8.69 453	255	11.30 547	9.99 947	10
51	8.69 654	253	8.69 708	254	11.30 292	9.99 946	9
52	8.69 907	252	8.69 962	252	11.30 038	9.99 946	8
53	8.70 159	250	8.70 214	251	11.29 786	9.99 945	7
54	8.70 409	249	8.70 465	249	11.29 535	9.99 944	6
55	8.70 658	247	8.70 714	248	11.29 286	9.99 944	5
56	8.70 905	246	8.70 962	246	11.29 038	9.99 943	4
57	8.71 151	244	8.71 208	245	11.28 792	9.99 942	3
58	8.71 395	243	8.71 453	244	11.28 547	9.99 942	2
59	8.71 638	242	8.71 697	243	11.28 303	9.99 941	1
60	8.71 880		8.71 940		11.28 060	9.99 940	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	360	355	350	345	340	335	330	325	320	315	310	305
2	72	71.0	70	69.0	68	67.0	66	65.0	64	63.0	62	61.0
3	108	106.5	105	103.5	102	100.5	99	97.5	96	94.5	93	91.5
4	144	142.0	140	138.0	136	134.0	132	130.0	128	126.0	124	122.0
5	180	177.5	175	172.5	170	167.5	165	162.5	160	157.5	155	152.5
6	216	213.0	210	207.0	204	201.0	198	195.0	192	189.0	186	183.0
7	252	248.5	245	241.5	238	234.5	231	227.5	224	220.5	217	213.5
8	288	284.0	280	276.0	272	268.0	264	260.0	256	252.0	248	244.0
9	324	319.5	315	310.5	306	301.5	297	292.5	288	283.5	279	274.5

	300	295	290	285	280	275	270	265	260	255	250	245
2	60	59.0	58	57.0	56	55.0	54	53.0	52	51.0	50	49.0
3	90	88.5	87	85.5	84	82.5	81	79.5	78	76.5	75	73.5
4	120	118.0	116	114.0	112	110.0	108	106.0	104	102.0	100	98.0
5	150	147.5	145	142.5	140	137.5	135	132.5	130	127.5	125	122.5
6	180	177.0	174	171.0	168	165.0	162	159.0	156	153.0	150	147.0
7	210	206.5	203	199.5	196	192.5	189	185.5	182	178.5	175	171.5
8	240	236.0	232	228.0	224	220.0	216	212.0	208	204.0	200	196.0
9	270	265.5	261	256.5	252	247.5	243	238.5	234	229.5	225	220.5

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	8.71 880	240	8.71 940	241	11.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	11.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	11.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	11.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	11.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	11.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	11.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	11.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	11.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	11.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	11.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	11.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	11.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	11.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	11.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	11.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	11.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	11.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	11.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	11.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	11.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	11.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	11.23 042	9.99 925	38
23	8.77 097	213	8.77 173	214	11.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	11.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	11.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	11.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	11.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	11.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	11.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	11.21 351	9.99 919	30
31	8.78 774	205	8.78 855	206	11.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	11.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	11.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	11.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	11.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	11.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	11.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	11.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	11.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	11.19 326	9.99 911	20
41	8.80 782	196	8.80 872	196	11.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	11.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	11.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	11.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	11.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	11.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	11.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	11.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	11.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	11.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	11.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	11.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	11.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	11.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	11.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	11.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	11.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	11.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	11.15 718	9.99 895	1
60	8.84 358		8.84 464		11.15 536	9.99 894	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	240	235	230
2	48	47.0	46
3	72	70.5	69
4	96	94.0	92
5	120	117.5	115
6	144	141.0	138
7	168	164.5	161
8	192	188.0	184
9	216	211.5	207
	225	**220**	**215**
2	45.0	44.0	43.0
3	67.5	66.0	64.5
4	90.0	88.0	86.0
5	112.5	110.0	107.5
6	135.0	132.0	129.0
7	157.5	154.0	150.5
8	180.0	176.0	172.0
9	202.5	198.0	193.5
	213	**211**	**208**
2	42.6	42.2	41.6
3	63.9	63.3	62.4
4	85.2	84.4	83.2
5	106.5	105.5	104.0
6	127.8	126.6	124.8
7	149.1	147.7	145.6
8	170.4	168.8	166.4
9	191.7	189.9	187.2
	206	**203**	**201**
2	41.2	40.6	40.2
3	61.8	60.9	60.3
4	82.4	81.2	80.4
5	103.0	101.5	100.5
6	123.6	121.8	120.6
7	144.2	142.1	140.7
8	164.8	162.4	160.8
9	185.4	182.7	180.9
	199	**197**	**195**
2	39.8	39.4	39.0
3	59.7	59.1	58.5
4	79.6	78.8	78.0
5	99.5	98.5	97.5
6	119.4	118.2	117.0
7	139.3	137.9	136.5
8	159.2	157.6	156.0
9	179.1	177.3	175.5
	193	**192**	**190**
2	38.6	38.4	38.0
3	57.9	57.6	57.0
4	77.2	76.8	76.0
5	96.5	96.0	95.0
6	115.8	115.2	114.0
7	135.1	134.4	133.0
8	154.4	153.6	152.0
9	173.7	172.8	171.0
	188	**186**	**184**
2	37.6	37.2	36.8
3	56.4	55.8	55.2
4	75.2	74.4	73.6
5	94.0	93.0	92.0
6	112.8	111.6	110.4
7	131.6	130.2	128.8
8	150.4	148.8	147.2
9	169.2	167.4	165.6
	183	**182**	**181**
2	36.6	36.4	36.2
3	54.9	54.6	54.3
4	73.2	72.8	72.4
5	91.5	91.0	90.5
6	109.8	109.2	108.6
7	128.1	127.4	126.7
8	146.4	145.6	144.8
9	164.7	163.8	162.9

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	8.84 358	181	8.84 464	182	11.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	11.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	11.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	11.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	11.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	11.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	11.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	11.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	11.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	11.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	11.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	11.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	11.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	11.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	11.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	11.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	11.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	11.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	11.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	11.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	11.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	11.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	11.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	11.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	11.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	11.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	11.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	11.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	11.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	11.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	11.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	11.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	11.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	11.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	11.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	11.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	11.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	11.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	11.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	11.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	11.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	11.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	11.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	11.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	11.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	11.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	11.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	11.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	11.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	11.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	11.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	11.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	11.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	11.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	11.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	11.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	11.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	11.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	11.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	11.05 951	9.99 836	1
60	8.94 030		8.94 195		11.05 805	9.99 834	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	181	180	179
2	36.2	36.0	35.8
3	54.3	54.0	53.7
4	72.4	72.0	71.6
5	90.5	90.0	89.5
6	108.6	108.0	107.4
7	126.7	126.0	125.3
8	144.8	144.0	143.2
9	162.9	162.0	161.1

	177	175	173
2	35.4	35.0	34.6
3	53.1	52.5	51.9
4	70.8	70.0	69.2
5	88.5	87.5	86.5
6	106.2	105.0	103.8
7	123.9	122.5	121.1
8	141.6	140.0	138.4
9	159.3	157.5	155.7

	171	170	169
2	34.2	34.0	33.8
3	51.3	51.0	50.7
4	68.4	68.0	67.6
5	85.5	85.0	84.5
6	102.6	102.0	101.4
7	119.7	119.0	118.3
8	136.8	136.0	135.2
9	153.9	153.0	152.1

	167	165	163
2	33.4	33.0	32.6
3	50.1	49.5	48.9
4	66.8	66.0	65.2
5	83.5	82.5	81.5
6	100.2	99.0	97.8
7	116.9	115.5	114.1
8	133.6	132.0	130.4
9	150.3	148.5	146.7

	161	160	159
2	32.2	32.0	31.8
3	48.3	48.0	47.7
4	64.4	64.0	63.6
5	80.5	80.0	79.5
6	96.6	96.0	95.4
7	112.7	112.0	111.3
8	128.8	128.0	127.2
9	144.9	144.0	143.1

	157	155	153
2	31.4	31.0	30.6
3	47.1	46.5	45.9
4	62.8	62.0	61.2
5	78.5	77.5	76.5
6	94.2	93.0	91.8
7	109.9	108.5	107.1
8	125.6	124.0	122.4
9	141.3	139.5	137.7

	151	150	149
2	30.2	30.0	29.8
3	45.3	45.0	44.7
4	60.4	60.0	59.6
5	75.5	75.0	74.5
6	90.6	90.0	89.4
7	105.7	105.0	104.3
8	120.8	120.0	119.2
9	135.9	135.0	134.1

	147	145	144
2	29.4	29.0	28.8
3	44.1	43.5	43.2
4	58.8	58.0	57.6
5	73.5	72.5	72.0
6	88.2	87.0	86.4
7	102.9	101.5	100.8
8	117.6	116.0	115.2
9	132.3	130.5	129.6

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	8.94 030	144	8.94 195	145	11.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	11.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	11.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	11.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	11.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	11.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	11.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	11.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	11.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	11.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	11.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	11.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	11.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	11.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	11.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	11.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	11.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	11.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	11.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	11.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	11.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	11.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	11.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	11.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	11.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	11.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	11.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	11.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	11.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	11.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	11.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	11.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	11.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	11.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	11.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	11.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	11.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	11.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	11.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	11.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	11.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	11.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	11.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	10.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	10.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	10.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	10.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	10.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	10.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	10.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	10.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	10.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	10.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	10.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	10.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	10.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	10.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	10.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	10.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	10.97 960	9.99 763	1
60	9.01 923		9.02 162		10.97 838	9.99 761	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	143	142	141
2	28.6	28.4	28.2
3	42.9	42.6	42.3
4	57.2	56.8	56.4
5	71.5	71.0	70.5
6	85.8	85.2	84.6
7	100.1	99.4	98.7
8	114.4	113.6	112.8
9	128.7	127.8	126.9
	140	**139**	**138**
2	28.0	27.8	27.6
3	42.0	41.7	41.4
4	56.0	55.6	55.2
5	70.0	69.5	69.0
6	84.0	83.4	82.8
7	98.0	97.3	96.6
8	112.0	111.2	110.4
9	126.0	125.1	124.2
	137	**136**	**135**
2	27.4	27.2	27.0
3	41.1	40.8	40.5
4	54.8	54.4	54.0
5	68.5	68.0	67.5
6	82.2	81.6	81.0
7	95.9	95.2	94.5
8	109.6	108.8	108.0
9	123.3	122.4	121.5
	134	**133**	**132**
2	26.8	26.6	26.4
3	40.2	39.9	39.6
4	53.6	53.2	52.8
5	67.0	66.5	66.0
6	80.4	79.8	79.2
7	93.8	93.1	92.4
8	107.2	106.4	105.6
9	120.6	119.7	118.8
	131	**130**	**129**
2	26.2	26.0	25.8
3	39.3	39.0	38.7
4	52.4	52.0	51.6
5	65.5	65.0	64.5
6	78.6	78.0	77.4
7	91.7	91.0	90.3
8	104.8	104.0	103.2
9	117.9	117.0	116.1
	128	**127**	**126**
2	25.6	25.4	25.2
3	38.4	38.1	37.8
4	51.2	50.8	50.4
5	64.0	63.5	63.0
6	76.8	76.2	75.6
7	89.6	88.9	88.2
8	102.4	101.6	100.8
9	115.2	114.3	113.4
	125	**124**	**123**
2	25.0	24.8	24.6
3	37.5	37.2	36.9
4	50.0	49.6	49.2
5	62.5	62.0	61.5
6	75.0	74.4	73.8
7	87.5	86.8	86.1
8	100.0	99.2	98.4
9	112.5	111.6	110.7
	122	**121**	**120**
2	24.4	24.2	24.0
3	36.6	36.3	36.0
4	48.8	48.4	48.0
5	61.0	60.5	60.0
6	73.2	72.6	72.0
7	85.4	84.7	84.0
8	97.6	96.8	96.0
9	109.8	108.9	108.0

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	9.01 923	120	9.02 162	121	10.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	10.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	10.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	10.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	10.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	10.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	10.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	10.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	10.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	10.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	10.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	10.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	10.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	10.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	10.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	10.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	10.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	10.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	10.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	10.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	10.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	10.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	10.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	10.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	10.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	10.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	10.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	10.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	10.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	10.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	10.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	10.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	10.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	10.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	10.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	10.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	10.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	10.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	10.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	10.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	10.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	10.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	10.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	10.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	10.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	10.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	10.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	10.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	10.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	10.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	10.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	10.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	10.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	10.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	10.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	10.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	10.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	10.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	10.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	10.91 190	9.99 677	1
60	9.08 589		9.08 914		10.91 086	9.99 675	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	121	120	119
2	24.2	24.0	23.8
3	36.3	36.0	35.7
4	48.4	48.0	47.6
5	60.5	60.0	59.5
6	72.6	72.0	71.4
7	84.7	84.0	83.3
8	96.8	96.0	95.2
9	108.9	108.0	107.1

	118	117	116
2	23.6	23.4	23.2
3	35.4	35.1	34.8
4	47.2	46.8	46.4
5	59.0	58.5	58.0
6	70.8	70.2	69.6
7	82.6	81.9	81.2
8	94.4	93.6	92.8
9	106.2	105.3	104.4

	115	114	113
2	23.0	22.8	22.6
3	34.5	34.2	33.9
4	46.0	45.6	45.2
5	57.5	57.0	56.5
6	69.0	68.4	67.8
7	80.5	79.8	79.1
8	92.0	91.2	90.4
9	103.5	102.6	101.7

	112	111	110
2	22.4	22.2	22.0
3	33.6	33.3	33.0
4	44.8	44.4	44.0
5	56.0	55.5	55.0
6	67.2	66.6	66.0
7	78.4	77.7	77.0
8	89.6	88.8	88.0
9	100.8	99.9	99.0

	109	108	107
2	21.8	21.6	21.4
3	32.7	32.4	32.1
4	43.6	43.2	42.8
5	54.5	54.0	53.5
6	65.4	64.8	64.2
7	76.3	75.6	74.9
8	87.2	86.4	85.6
9	98.1	97.2	96.3

	106	105	104
2	21.2	21.0	20.8
3	31.8	31.5	31.2
4	42.4	42.0	41.6
5	53.0	52.5	52.0
6	63.6	63.0	62.4
7	74.2	73.5	72.8
8	84.8	84.0	83.2
9	95.4	94.5	93.6

From the top:

For **6°+** or **186°+**, read as printed; for **96°+** or **276°+**, read co-function.

From the bottom:

For **83°+** or **263°+**, read as printed; for **173°+** or **353°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	9.08 589	103	9.08 914	105	10.91 086	9.99 675	60
1	9.08 692	103	9.09 019	104	10.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	10.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	10.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	10.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	10.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	10.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	10.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	10.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	10.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	10.90 053	9.99 659	50
11	9.09 707	100	9.10 049	101	10.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	10.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	10.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	10.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	10.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	10.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	10.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	10.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	10.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	10.89 044	9.99 643	40
21	9.10 697	98	9.11 056	99	10.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	10.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	10.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	10.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	10.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	10.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	10.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	10.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	10.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	10.88 057	9.99 627	30
31	9.11 666	95	9.12 040	98	10.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	10.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	10.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	10.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	10.87 572	9.99 618	25
36	9.12 142	94	9.12 525	96	10.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	10.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	10.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	10.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	10.87 091	9.99 610	20
41	9.12 612	94	9.13 004	95	10.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	10.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	10.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	10.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	10.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	10.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	10.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	10.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	10.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	10.86 146	9.99 593	10
51	9.13 539	91	9.13 948	93	10.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	10.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	10.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	10.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	10.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	10.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	10.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	10.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	10.85 312	9.99 577	1
60	9.14 356		9.14 780		10.85 220	9.99 575	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	105	104	103
2	21.0	20.8	20.6
3	31.5	31.2	30.9
4	42.0	41.6	41.2
5	52.5	52.0	51.5
6	63.0	62.4	61.8
7	73.5	72.8	72.1
8	84.0	83.2	82.4
9	94.5	93.6	92.7

	102	101	99
2	20.4	20.2	19.8
3	30.6	30.3	29.7
4	40.8	40.4	39.6
5	51.0	50.5	49.5
6	61.2	60.6	59.4
7	71.4	70.7	69.3
8	81.6	80.8	79.2
9	91.8	90.9	89.1

	98	97	96
2	19.6	19.4	19.2
3	29.4	29.1	28.8
4	39.2	38.8	38.4
5	49.0	48.5	48.0
6	58.8	58.2	57.6
7	68.6	67.9	67.2
8	78.4	77.6	76.8
9	88.2	87.3	86.4

	95	94	93
2	19.0	18.8	18.6
3	28.5	28.2	27.9
4	38.0	37.6	37.2
5	47.5	47.0	46.5
6	57.0	56.4	55.8
7	66.5	65.8	65.1
8	76.0	75.2	74.4
9	85.5	84.6	83.7

	92	91	90
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

From the top:

For **7°+** or **187°+**, read as printed; for **97°+** or **277°+**, read co-function.

From the bottom:

For **82°+** or **262°+**, read as printed; for **172°+** or **352°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	
0	9.14 356	89	9.14 780	92	10.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	10.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	10.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	10.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	10.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	10.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	10.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	10.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	10.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	10.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	10.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	10.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	10.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	10.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	10.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	10.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	10.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	10.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	10.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	10.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	10.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	10.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	10.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	10.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	10.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	10.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	10.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	10.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	10.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	10.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	10.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	10.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	10.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	10.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	10.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	10.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	10.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	10.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	10.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	10.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	10.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	10.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	10.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	10.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	10.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	10.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	10.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	10.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	10.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	10.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	10.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	10.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	10.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	10.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	10.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	10.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	10.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	10.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	10.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	10.80 111	9.99 464	1
60	9.19 433		9.19 971		10.80 029	9.99 462	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	′

Prop. Pts.

	92	91	90
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

	89	88	87
2	17.8	17.6	17.4
3	26.7	26.4	26.1
4	35.6	35.2	34.8
5	44.5	44.0	43.5
6	53.4	52.8	52.2
7	62.3	61.6	60.9
8	71.2	70.4	69.6
9	80.1	79.2	78.3

	86	85	84
2	17.2	17.0	16.8
3	25.8	25.5	25.2
4	34.4	34.0	33.6
5	43.0	42.5	42.0
6	51.6	51.0	50.4
7	60.2	59.5	58.8
8	68.8	68.0	67.2
9	77.4	76.5	75.6

	83	82	81
2	16.6	16.4	16.2
3	24.9	24.6	24.3
4	33.2	32.8	32.4
5	41.5	41.0	40.5
6	49.8	49.2	48.6
7	58.1	57.4	56.7
8	66.4	65.6	64.8
9	74.7	73.8	72.9

From the top:

For **8°+** or **188°+**, read as printed; for **98°+** or **278°+**, read co-function.

From the bottom:

For **81°+** or **261°+**, read as printed; for **171°+** or **351°+**, read co-function.

81° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	cd	L Ctn	L Cos	
0	9.19 433	80	9.19 971	82	10.80 029	9.99 462	**60**
1	9.19 513	79	9.20 053	81	10.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	10.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	10.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	10.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	10.79 622	9.99 452	**55**
6	9.19 909	79	9.20 459	81	10.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	10.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	10.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	10.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	10.79 218	9.99 442	**50**
11	9.20 302	78	9.20 862	80	10.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	10.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	10.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	10.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	10.78 818	9.99 432	**45**
16	9.20 691	77	9.21 261	80	10.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	10.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	10.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	10.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	10.78 422	9.99 421	**40**
21	9.21 076	77	9.21 657	79	10.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	10.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	10.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	10.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	10.78 029	9.99 411	**35**
26	9.21 458	76	9.22 049	78	10.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	10.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	10.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	10.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	10.77 639	9.99 400	**30**
31	9.21 836	76	9.22 438	78	10.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	10.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	10.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	10.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	10.77 253	9.99 390	**25**
36	9.22 211	75	9.22 824	77	10.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	10.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	10.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	10.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	10.76 870	9.99 379	**20**
41	9.22 583	74	9.23 206	77	10.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	10.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	10.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	10.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	10.76 490	9.99 368	**15**
46	9.22 952	73	9.23 586	75	10.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	10.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	10.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	10.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	10.76 113	9.99 357	**10**
51	9.23 317	73	9.23 962	75	10.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	10.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	10.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	10.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	10.75 739	9.99 346	**5**
56	9.23 679	73	9.24 335	75	10.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	10.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	10.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	10.75 442	9.99 337	1
60	9.23 967		9.24 632		10.75 368	9.99 335	**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	′

Prop. Pts.

	82	81	80
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

	79	78	77
2	15.8	15.6	15.4
3	23.7	23.4	23.1
4	31.6	31.2	30.8
5	39.5	39.0	38.5
6	47.4	46.8	46.2
7	55.3	54.6	53.9
8	63.2	62.4	61.6
9	71.1	70.2	69.3

	76	75	74
2	15.2	15.0	14.8
3	22.8	22.5	22.2
4	30.4	30.0	29.6
5	38.0	37.5	37.0
6	45.6	45.0	44.4
7	53.2	52.5	51.8
8	60.8	60.0	59.2
9	68.4	67.5	66.6

	73	72	71
2	14.6	14.4	14.2
3	21.9	21.6	21.3
4	29.2	28.8	28.4
5	36.5	36.0	35.5
6	43.8	43.2	42.6
7	51.1	50.4	49.7
8	58.4	57.6	56.8
9	65.7	64.8	63.9

From the top:

For **9°+**, or **189°+**, read as printed; for **99°+** or **279°+**, read co-function.

From the bottom:

For **80°+** or **260°+**, read as printed; for **170°+** or **350°+**, read co-function.

Prop. Pts.

80° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.23 967	72	9.24 632	74	10.75 368	9.99 335	2	60
1	9.24 039	71	9.24 706	73	10.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	74	10.75 221	9.99 331	3	58
3	9.24 181	72	9.24 853	73	10.75 147	9.99 328	2	57
4	9.24 253	71	9.24 926	74	10.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	73	10.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	10.74 927	9.99 322	3	54
7	9.24 466	70	9.25 146	73	10.74 854	9.99 319	2	53
8	9.24 536	71	9.25 219	73	10.74 781	9.99 317	2	52
9	9.24 607	70	9.25 292	73	10.74 708	9.99 315	2	51
10	9.24 677	71	9.25 365	72	10.74 635	9.99 313	3	50
11	9.24 748	70	9.25 437	73	10.74 563	9.99 310	2	49
12	9.24 818	70	9.25 510	72	10.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	73	10.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	72	10.74 345	9.99 304	3	46
15	9.25 028	70	9.25 727	72	10.74 273	9.99 301	2	45
16	9.25 098	70	9.25 799	72	10.74 201	9.99 299	2	44
17	9.25 168	69	9.25 871	72	10.74 129	9.99 297	3	43
18	9.25 237	70	9.25 943	72	10.74 057	9.99 294	2	42
19	9.25 307	69	9.26 015	71	10.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	72	10.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	71	10.73 842	9.99 288	3	39
22	9.25 514	69	9.26 229	72	10.73 771	9.99 285	2	38
23	9.25 583	69	9.26 301	71	10.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	10.73 628	9.99 281	3	36
25	9.25 721	69	9.26 443	71	10.73 557	9.99 278	2	35
26	9.25 790	68	9.26 514	71	10.73 486	9.99 276	2	34
27	9.25 858	69	9.26 585	70	10.73 415	9.99 274	3	33
28	9.25 927	68	9.26 655	71	10.73 345	9.99 271	2	32
29	9.25 995	68	9.26 726	71	10.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	70	10.73 203	9.99 267	3	30
31	9.26 131	68	9.26 867	70	10.73 133	9.99 264	2	29
32	9.26 199	68	9.26 937	71	10.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	70	10.72 992	9.99 260	3	27
34	9.26 335	68	9.27 078	70	10.72 922	9.99 257	2	26
35	9.26 403	67	9.27 148	70	10.72 852	9.99 255	3	25
36	9.26 470	68	9.27 218	70	10.72 782	9.99 252	2	24
37	9.26 538	67	9.27 288	69	10.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	70	10.72 643	9.99 248	3	22
39	9.26 672	67	9.27 427	69	10.72 573	9.99 245	2	21
40	9.26 739	67	9.27 496	70	10.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	69	10.72 434	9.99 241	3	19
42	9.26 873	67	9.27 635	69	10.72 365	9.99 238	2	18
43	9.26 940	67	9.27 704	69	10.72 296	9.99 236	3	17
44	9.27 007	66	9.27 773	69	10.72 227	9.99 233	2	16
45	9.27 073	67	9.27 842	69	10.72 158	9.99 231	2	15
46	9.27 140	66	9.27 911	69	10.72 089	9.99 229	3	14
47	9.27 206	67	9.27 980	69	10.72 020	9.99 226	2	13
48	9.27 273	66	9.28 049	68	10.71 951	9.99 224	3	12
49	9.27 339	66	9.28 117	69	10.71 883	9.99 221	2	11
50	9.27 405	66	9.28 186	68	10.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	69	10.71 746	9.99 217	3	9
52	9.27 537	65	9.28 323	68	10.71 677	9.99 214	2	8
53	9.27 602	66	9.28 391	68	10.71 609	9.99 212	3	7
54	9.27 668	66	9.28 459	68	10.71 541	9.99 209	2	6
55	9.27 734	65	9.28 527	68	10.71 473	9.99 207	3	5
56	9.27 799	65	9.28 595	67	10.71 405	9.99 204	2	4
57	9.27 864	66	9.28 662	68	10.71 338	9.99 202	2	3
58	9.27 930	65	9.28 730	68	10.71 270	9.99 200	3	2
59	9.27 995	65	9.28 798	67	10.71 202	9.99 197	2	1
60	9.28 060		9.28 865		10.71 135	9.99 195		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	74	73	72
2	14.8	14.6	14.4
3	22.2	21.9	21.6
4	29.6	29.2	28.8
5	37.0	36.5	36.0
6	44.4	43.8	43.2
7	51.8	51.1	50.4
8	59.2	58.4	57.6
9	66.6	65.7	64.8

	71	70	69
2	14.2	14.0	13.8
3	21.3	21.0	20.7
4	28.4	28.0	27.6
5	35.5	35.0	34.5
6	42.6	42.0	41.4
7	49.7	49.0	48.3
8	56.8	56.0	55.2
9	63.9	63.0	62.1

	68	67	66
2	13.6	13.4	13.2
3	20.4	20.1	19.8
4	27.2	26.8	26.4
5	34.0	33.5	33.0
6	40.8	40.2	39.6
7	47.6	46.9	46.2
8	54.4	53.6	52.8
9	61.2	60.3	59.4

	65	3	2
2	13.0	0.6	0.4
3	19.5	0.9	0.6
4	26.0	1.2	0.8
5	32.5	1.5	1.0
6	39.0	1.8	1.2
7	45.5	2.1	1.4
8	52.0	2.4	1.6
9	58.5	2.7	1.8

From the top:
For **10°+** or **190°+**, read as printed; for **100°+** or **280°+**, read co-function.

From the bottom:
For **79°+** or **259°+**, read as printed; for **169°+** or **349°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.28 060	65	9.28 865	68	10.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	10.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	10.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	10.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	10.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	10.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	10.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	10.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	10.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	10.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	10.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	10.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	10.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	10.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	10.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	10.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	10.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	10.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	10.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	10.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	10.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	10.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	10.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	10.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	10.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	10.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	10.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	10.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	10.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	10.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	10.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	10.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	10.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	10.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	10.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	10.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	10.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	10.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	10.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	10.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	10.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	10.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	10.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	10.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	10.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	10.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	10.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	10.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	10.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	10.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	10.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	10.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	10.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	10.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	10.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	10.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	10.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	10.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	10.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	10.67 315	9.99 043	3	1
60	9.31 788		9.32 747		10.67 253	9.99 040		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	68	67	66
2	13.6	13.4	13.2
3	20.4	20.1	19.8
4	27.2	26.8	26.4
5	34.0	33.5	33.0
6	40.8	40.2	39.6
7	47.6	46.9	46.2
8	54.4	53.6	52.8
9	61.2	60.3	59.4

	65	64	63
2	13.0	12.8	12.6
3	19.5	19.2	18.9
4	26.0	25.6	25.2
5	32.5	32.0	31.5
6	39.0	38.4	37.8
7	45.5	44.8	44.1
8	52.0	51.2	50.4
9	58.5	57.6	56.7

	62	61	60
2	12.4	12.2	12.0
3	18.6	18.3	18.0
4	24.8	24.4	24.0
5	31.0	30.5	30.0
6	37.2	36.6	36.0
7	43.4	42.7	42.0
8	49.6	48.8	48.0
9	55.8	54.9	54.0

	59	3	2
2	11.8	0.6	0.4
3	17.7	0.9	0.6
4	23.6	1.2	0.8
5	29.5	1.5	1.0
6	35.4	1.8	1.2
7	41.3	2.1	1.4
8	47.2	2.4	1.6
9	53.1	2.7	1.8

From the top:

For **11°+** or **191°+**, read as printed; for **101°+** or **281°+**, read co-function.

From the bottom:

For **78°+** or **258°+**, read as printed; for **168°+** or **348°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.31 788	59	9.32 747	63	10.67 253	9.99 040	2	**60**
1	9.31 847	60	9.32 810	62	10.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	10.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	10.67 067	9.99 032	2	57
4	9.32 025	59	9.32 995	62	10.67 005	9.99 030	3	56
5	9.32 084	59	9.33 057	62	10.66 943	9.99 027	3	**55**
6	9.32 143	59	9.33 119	61	10.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	10.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	10.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	10.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365	61	10.66 635	9.99 013	2	**50**
11	9.32 437	58	9.33 426	61	10.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	10.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	10.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	10.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	10.66 330	9.99 000	3	**45**
16	9.32 728	58	9.33 731	61	10.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	10.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	10.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	10.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	10.66 026	9.98 986	3	**40**
21	9.33 018	57	9.34 034	61	10.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095	60	10.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	10.65 845	9.98 978	3	37
24	9.33 190	58	9.34 215	61	10.65 785	9.98 975	3	36
25	9.33 248	57	9.34 276	60	10.65 724	9.98 972	3	**35**
26	9.33 305	57	9.34 336	60	10.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	10.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	10.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	10.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	10.65 424	9.98 958	3	**30**
31	9.33 591	56	9.34 635	60	10.65 365	9.98 955	2	29
32	9.33 647	57	9.34 695	60	10.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755	59	10.65 245	9.98 950	3	27
34	9.33 761	57	9.34 814	60	10.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	10.65 126	9.98 944	3	**25**
36	9.33 874	57	9.34 933	59	10.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	10.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	10.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	10.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	10.64 830	9.98 930	3	**20**
41	9.34 156	56	9.35 229	59	10.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	10.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	10.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	10.64 595	9.98 919	3	16
45	9.34 380	56	9.35 464	59	10.64 536	9.98 916	3	**15**
46	9.34 436	55	9.35 523	58	10.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	10.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	10.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	10.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	10.64 243	9.98 901	3	**10**
51	9.34 713	56	9.35 815	58	10.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	10.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	10.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	10.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	10.63 953	9.98 887	3	**5**
56	9.34 989	55	9.36 105	58	10.63 895	9.98 884	3	4
57	9.35 044	55	9.36 163	58	10.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	10.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	10.63 721	9.98 875	3	1
60	9.35 209		9.36 336		10.63 664	9.98 872		**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	63	62	61
2	12.6	12.4	12.2
3	18.9	18.6	18.3
4	25.2	24.8	24.4
5	31.5	31.0	30.5
6	37.8	37.2	36.6
7	44.1	43.4	42.7
8	50.4	49.6	48.8
9	56.7	55.8	54.9

	60	59	58
2	12.0	11.8	11.6
3	18.0	17.7	17.4
4	24.0	23.6	23.2
5	30.0	29.5	29.0
6	36.0	35.4	34.8
7	42.0	41.3	40.6
8	48.0	47.2	46.4
9	54.0	53.1	52.2

	57	56
2	11.4	11.2
3	17.1	16.8
4	22.8	22.4
5	28.5	28.0
6	34.2	33.6
7	39.9	39.2
8	45.6	44.8
9	51.3	50.4

	55	3	2
2	11.0	0.6	0.4
3	16.5	0.9	0.6
4	22.0	1.2	0.8
5	27.5	1.5	1.0
6	33.0	1.8	1.2
7	38.5	2.1	1.4
8	44.0	2.4	1.6
9	49.5	2.7	1.8

From the top:

For **12°+** or **192°+**, read as printed; for **102°+** or **282°+**, read co-function.

From the bottom:

For **77°+** or **257°+**, read as printed; for **167°+** or **347°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	′
0	9.35 209	54	9.36 336	58	10.63 664	9.98 872	3	60
1	9.35 263	55	9.36 394	58	10.63 606	9.98 869	2	59
2	9.35 318	55	9.36 452	57	10.63 548	9.98 867	3	58
3	9.35 373	54	9.36 509	57	10.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	58	10.63 434	9.98 861	3	56
5	9.35 481	55	9.36 624	57	10.63 376	9.98 858	3	55
6	9.35 536	54	9.36 681	57	10.63 319	9.98 855	3	54
7	9.35 590	54	9.36 738	57	10.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	10.63 205	9.98 849	3	52
9	9.35 698	54	9.36 852	57	10.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	10.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	10.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	10.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	10.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	56	10.62 863	9.98 831	3	46
15	9.36 022	53	9.37 193	57	10.62 807	9.98 828	3	45
16	9.36 075	54	9.37 250	56	10.62 750	9.98 825	3	44
17	9.36 129	53	9.37 306	57	10.62 694	9.98 822	3	43
18	9.36 182	54	9.37 363	56	10.62 637	9.98 819	3	42
19	9.36 236	53	9.37 419	57	10.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	56	10.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	10.62 468	9.98 810	3	39
22	9.36 395	54	9.37 588	56	10.62 412	9.98 807	3	38
23	9.36 449	53	9.37 644	56	10.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	10.62 300	9.98 801	3	36
25	9.36 555	53	9.37 756	56	10.62 244	9.98 798	3	35
26	9.36 608	52	9.37 812	56	10.62 188	9.98 795	3	34
27	9.36 660	53	9.37 868	56	10.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	10.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	55	10.62 020	9.98 786	3	31
30	9.36 819	52	9.38 035	56	10.61 965	9.98 783	3	30
31	9.36 871	53	9.38 091	56	10.61 909	9.98 780	3	29
32	9.36 924	52	9.38 147	55	10.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	10.61 798	9.98 774	3	27
34	9.37 028	53	9.38 257	56	10.61 743	9.98 771	3	26
35	9.37 081	52	9.38 313	55	10.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	10.61 632	9.98 765	3	24
37	9.37 185	52	9.38 423	56	10.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	55	10.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	10.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	10.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	10.61 356	9.98 750	4	19
42	9.37 445	52	9.38 699	55	10.61 301	9.98 746	3	18
43	9.37 497	52	9.38 754	54	10.61 246	9.98 743	3	17
44	9.37 549	51	9.38 808	55	10.61 192	9.98 740	3	16
45	9.37 600	52	9.38 863	55	10.61 137	9.98 737	3	15
46	9.37 652	51	9.38 918	54	10.61 082	9.98 734	3	14
47	9.37 703	52	9.38 972	55	10.61 028	9.98 731	3	13
48	9.37 755	51	9.39 027	55	10.60 973	9.98 728	3	12
49	9.37 806	52	9.39 082	54	10.60 918	9.98 725	3	11
50	9.37 858	51	9.39 136	54	10.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	55	10.60 810	9.98 719	4	9
52	9.37 960	51	9.39 245	54	10.60 755	9.98 715	3	8
53	9.38 011	51	9.39 299	54	10.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	10.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	10.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	10.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	10.60 485	9.98 700	3	3
58	9.38 266	51	9.39 569	54	10.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	10.60 377	9.98 694	4	1
60	9.38 368		9.39 677		10.60 323	9.98 690		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	58	57	56
2	11.6	11.4	11.2
3	17.4	17.1	16.8
4	23.2	22.8	22.4
5	29.0	28.5	28.0
6	34.8	34.2	33.6
7	40.6	39.9	39.2
8	46.4	45.6	44.8
9	52.2	51.3	50.4

	55	54	53
2	11.0	10.8	10.6
3	16.5	16.2	15.9
4	22.0	21.6	21.2
5	27.5	27.0	26.5
6	33.0	32.4	31.8
7	38.5	37.8	37.1
8	44.0	43.2	42.4
9	49.5	48.6	47.7

	52	51
2	10.4	10.2
3	15.6	15.3
4	20.8	20.4
5	26.0	25.5
6	31.2	30.6
7	36.4	35.7
8	41.6	40.8
9	46.8	45.9

	4	3	2
2	0.8	0.6	0.4
3	1.2	0.9	0.6
4	1.6	1.2	0.8
5	2.0	1.5	1.0
6	2.4	1.8	1.2
7	2.8	2.1	1.4
8	3.2	2.4	1.6
9	3.6	2.7	1.8

From the top:

For **13°+** or **193°+**, read as printed; for **103°+** or **283°+**, read co-function.

From the bottom:

For **76°+** or **256°+**, read as printed; for **166°+** or **346°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.38 368	50	9.39 677	54	10.60 323	9.98 690	3	60
1	9.38 418	51	9.39 731	54	10.60 269	9.98 687	3	59
2	9.38 469	50	9.39 785	53	10.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	10.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	10.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	10.60 055	9.98 675	4	55
6	9.38 670	51	9.39 999	53	10.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	10.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	10.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	10.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	10.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	53	10.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	10.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	10.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425	53	10.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	10.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	10.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	10.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	10.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	10.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	10.59 258	9.98 627	4	40
21	9.39 418	49	9.40 795	52	10.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	10.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	10.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	10.59 048	9.98 614	4	36
25	9.39 615	49	9.41 005	52	10.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	10.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	10.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	10.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	10.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	10.58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	10.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	10.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	10.58 578	9.98 584	3	27
34	9.40 055	48	9.41 474	52	10.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	10.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	10.58 422	9.98 574	3	24
37	9.40 200	49	9.41 629	52	10.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	10.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	10.58 267	9.98 565	4	21
40	9.40 346	48	9.41 784	52	10.58 216	9.98 561	3	20
41	9.40 394	48	9.41 836	51	10.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	10.58 113	9.98 555	4	18
43	9.40 490	48	9.41 939	51	10.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	10.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	10.57 959	9.98 545	4	15
46	9.40 634	48	9.42 093	51	10.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	10.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	10.57 805	9.98 535	4	12
49	9.40 778	47	9.42 246	51	10.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	10.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	10.57 652	9.98 525	4	9
52	9.40 921	47	9.42 399	51	10.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	10.57 550	9.98 518	3	7
54	9.41 016	47	9.42 501	51	10.57 499	9.98 515	4	6
55	9.41 063	48	9.42 552	51	10.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	10.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	10.57 347	9.98 505	4	3
58	9.41 205	47	9.42 704	51	10.57 296	9.98 501	3	2
59	9.41 252	48	9.42 755	50	10.57 245	9.98 498	4	1
60	9.41 300		9.42 805		10.57 195	9.98 494		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	54	53	52
2	10.8	10.6	10.4
3	16.2	15.9	15.6
4	21.6	21.2	20.8
5	27.0	26.5	26.0
6	32.4	31.8	31.2
7	37.8	37.1	36.4
8	43.2	42.4	41.6
9	48.6	47.7	46.8

	51	50	49
2	10.2	10.0	9.8
3	15.3	15.0	14.7
4	20.4	20.0	19.6
5	25.5	25.0	24.5
6	30.6	30.0	29.4
7	35.7	35.0	34.3
8	40.8	40.0	39.2
9	45.9	45.0	44.1

	48	47
2	9.6	9.4
3	14.4	14.1
4	19.2	18.8
5	24.0	23.5
6	28.8	28.2
7	33.6	32.9
8	38.4	37.6
9	43.2	42.3

	4	3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

From the top:

For 14°+ or 194°+, read as printed; for 104°+ or 284°+, read co-function.

From the bottom:

For 75°+ or 255°+, read as printed; for 165°+ or 345°+, read co-function.

Prop. Pts.

75° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.41 300	47	9.42 805	51	10.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	10.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	10.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	10.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	10.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	10.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	10.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	10.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	10.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	10.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	10.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	10.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	10.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	10.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	10.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	10.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	10.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	10.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	10.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	10.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	10.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	10.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	10.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	10.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	10.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	10.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	10.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	10.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	10.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	10.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	10.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	10.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	10.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	10.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	10.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	10.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	10.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	10.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	10.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	10.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	10.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	10.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	10.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	10.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	10.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	10.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	10.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	10.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	10.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	10.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	10.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	10.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	10.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	10.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	10.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	10.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	10.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	10.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	10.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	10.54 298	9.98 288	4	1
60	9.44 034		9.45 750		10.54 250	9.98 284		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	51	50	49
2	10.2	10.0	9.8
3	15.3	15.0	14.7
4	20.4	20.0	19.6
5	25.5	25.0	24.5
6	30.6	30.0	29.4
7	35.7	35.0	34.3
8	40.8	40.0	39.2
9	45.9	45.0	44.1

	48	47	46
2	9.6	9.4	9.2
3	14.4	14.1	13.8
4	19.2	18.8	18.4
5	24.0	23.5	23.0
6	28.8	28.2	27.6
7	33.6	32.9	32.2
8	38.4	37.6	36.8
9	43.2	42.3	41.4

	45	44
2	9.0	8.8
3	13.5	13.2
4	18.0	17.6
5	22.5	22.0
6	27.0	26.4
7	31.5	30.8
8	36.0	35.2
9	40.5	39.6

	4	3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

From the top:

For **15°+** or **195°+**, read as printed; for **105°+** or **285°+**, read co-function.

From the bottom:

For **74°+** or **254°+**, read as printed; for **164°+** or **344°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.44 034	44	9.45 750	47	10.54 250	9.98 284	3	**60**
1	9.44 078	44	9.45 797	48	10.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	10.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	10.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	10.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	10.54 013	9.98 266	4	**55**
6	9.44 297	44	9.46 035	47	10.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	10.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	10.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	10.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	10.53 776	9.98 248	4	**50**
11	9.44 516	43	9.46 271	48	10.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	10.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	10.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	10.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	10.53 540	9.98 229	3	**45**
16	9.44 733	43	9.46 507	47	10.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	10.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	10.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	10.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	10.53 306	9.98 211	4	**40**
21	9.44 948	44	9.46 741	47	10.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	10.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	10.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	10.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	10.53 072	9.98 192	3	**35**
26	9.45 163	43	9.46 975	46	10.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	10.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	10.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	10.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	10.52 840	9.98 174	4	**30**
31	9.45 377	42	9.47 207	46	10.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	10.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	10.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	10.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	10.52 608	9.98 155	4	**25**
36	9.45 589	43	9.47 438	46	10.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	10.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	10.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	10.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	10.52 378	9.98 136	4	**20**
41	9.45 801	42	9.47 668	46	10.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	10.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	10.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	10.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	10.52 148	9.98 117	4	**15**
46	9.46 011	42	9.47 897	46	10.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	10.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	10.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	10.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	10.51 920	9.98 098	4	**10**
51	9.46 220	42	9.48 126	45	10.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	10.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	10.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	10.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	10.51 693	9.98 079	4	**5**
56	9.46 428	41	9.48 353	45	10.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	10.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	10.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	10.51 511	9.98 063	3	1
60	9.46 594		9.48 534		10.51 466	9.98 060		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	48	47	46
2	9.6	9.4	9.2
3	14.4	14.1	13.8
4	19.2	18.8	18.4
5	24.0	23.5	23.0
6	28.8	28.2	27.6
7	33.6	32.9	32.2
8	38.4	37.6	36.8
9	43.2	42.3	41.4

	45	44	43
2	9.0	8.8	8.6
3	13.5	13.2	12.9
4	18.0	17.6	17.2
5	22.5	22.0	21.5
6	27.0	26.4	25.8
7	31.5	30.8	30.1
8	36.0	35.2	34.4
9	40.5	39.6	38.7

	42	41
2	8.4	8.2
3	12.6	12.3
4	16.8	16.4
5	21.0	20.5
6	25.2	24.6
7	29.4	28.7
8	33.6	32.8
9	37.8	36.9

	4	3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

From the top:

For **16°+** or **196°+**, read as printed; for **106°+** or **286°+**, read co-function.

From the bottom:

For **73°+** or **253°+**, read as printed; for **163°+** or **343°+**, read co-function.

Prop. Pts.

73° — Logarithms of Trigonometric Functions

'	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.46 594	41	9.48 534	45	10.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	10.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	10.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	10.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	10.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	10.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	10.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	10.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	10.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	10.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	10.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	10.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	10.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	10.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	10.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	10.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	10.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	10.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	10.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	10.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	10.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	10.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	10.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	10.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	10.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	10.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	10.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	10.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	10.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	10.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	10.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	10.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	10.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	10.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	10.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	10.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	10.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	10.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	10.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	10.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	10.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	10.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	10.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	10.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	10.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	10.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	10.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	10.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	10.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	10.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	10.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	10.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	10.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	10.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	10.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	10.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	10.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	10.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	10.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	10.48 865	9.97 825	4	1
60	9.48 998		9.51 178		10.48 822	9.97 821		0
	L Cos	**d**	**L Ctn**	**cd**	**L Tan**	**L Sin**	**d**	**'**

Prop. Pts.

	45	44	43
2	9.0	8.8	8.6
3	13.5	13.2	12.9
4	18.0	17.6	17.2
5	22.5	22.0	21.5
6	27.0	26.4	25.8
7	31.5	30.8	30.1
8	36.0	35.2	34.4
9	40.5	39.6	38.7

	42	41	40
2	8.4	8.2	8.0
3	12.6	12.3	12.0
4	16.8	16.4	16.0
5	21.0	20.5	20.0
6	25.2	24.6	24.0
7	29.4	28.7	28.0
8	33.6	32.8	32.0
9	37.8	36.9	36.0

	39	5
2	7.8	1.0
3	11.7	1.5
4	15.6	2.0
5	19.5	2.5
6	23.4	3.0
7	27.3	3.5
8	31.2	4.0
9	35.1	4.5

	4	3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

From the top:

For 17°+ or 197°+, read as printed; for 107°+ or 287°+, read co-function.

From the bottom:

For 72°+ or 252°+, read as printed; for 162°+ or 342°+, read co-function.

Prop. Pts.

72° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.48 998	39	9.51 178	43	10.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	10.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	10.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	10.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	10.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	10.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	10.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	10.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	10.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	10.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	10.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	10.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	10.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	10.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	10.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	10.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	10.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	10.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	10.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	10.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	10.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	10.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	10.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	10.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	10.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	10.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	10.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	10.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	10.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	10.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	10.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	10.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	10.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	10.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	10.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	10.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	10.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	10.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	10.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	10.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	10.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	10.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	10.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	10.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	10.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	10.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	10.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	10.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	10.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	10.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	10.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	10.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	10.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	10.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	10.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	10.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	10.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	10.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	10.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	10.46 344	9.97 571	4	1
60	9.51 264		9.53 697		10.46 303	9.97 567		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	43	42	41
2	8.6	8.4	8.2
3	12.9	12.6	12.3
4	17.2	16.8	16.4
5	21.5	21.0	20.5
6	25.8	25.2	24.6
7	30.1	29.4	28.7
8	34.4	33.6	32.8
9	38.7	37.8	36.9

	39	38	37
2	7.8	7.6	7.4
3	11.7	11.4	11.1
4	15.6	15.2	14.8
5	19.5	19.0	18.5
6	23.4	22.8	22.2
7	27.3	26.6	25.9
8	31.2	30.4	29.6
9	35.1	34.2	33.3

	36	5	4
2	7.2	1.0	0.8
3	10.8	1.5	1.2
4	14.4	2.0	1.6
5	18.0	2.5	2.0
6	21.6	3.0	2.4
7	25.2	3.5	2.8
8	28.8	4.0	3.2
9	32.4	4.5	3.6

From the top:
For 18°+ or 198°+, read as printed; for 108°+ or 288°+, read co-function.

From the bottom:
For 71°+ or 251°+, read as printed; for 161°+ or 341°+, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	′
0	9.51 264	37	9.53 697	41	10.46 303	9.97 567	4	**60**
1	9.51 301	37	9.53 738	41	10.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	10.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	10.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	10.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	10.46 098	9.97 545	4	**55**
6	9.51 484	36	9.53 943	41	10.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	10.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	10.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	10.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	10.45 894	9.97 523	4	**50**
11	9.51 666	36	9.54 147	40	10.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	10.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	10.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	10.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	10.45 691	9.97 501	4	**45**
16	9.51 847	36	9.54 350	40	10.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	10.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	10.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	10.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	10.45 488	9.97 479	4	**40**
21	9.52 027	36	9.54 552	41	10.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	10.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	10.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	10.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	10.45 286	9.97 457	4	**35**
26	9.52 207	35	9.54 754	40	10.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	10.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	10.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	10.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	10.45 085	9.97 435	5	**30**
31	9.52 385	36	9.54 955	40	10.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	10.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	10.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	10.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	10.44 885	9.97 412	4	**25**
36	9.52 563	35	9.55 155	40	10.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	10.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	10.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	10.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	10.44 685	9.97 390	5	**20**
41	9.52 740	35	9.55 355	40	10.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	10.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	10.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	10.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	10.44 486	9.97 367	4	**15**
46	9.52 916	35	9.55 554	39	10.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	10.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	10.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	10.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	10.44 288	9.97 344	4	**10**
51	9.53 092	34	9.55 752	39	10.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	10.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	10.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	10.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	10.44 090	9.97 322	5	**5**
56	9.53 266	35	9.55 949	40	10.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	10.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	10.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	10.43 933	9.97 303	4	1
60	9.53 405		9.56 107		10.43 893	9.97 299		**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	41	40	39
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	37	36	35
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	34	5	4
2	6.8	1.0	0.8
3	10.2	1.5	1.2
4	13.6	2.0	1.6
5	17.0	2.5	2.0
6	20.4	3.0	2.4
7	23.8	3.5	2.8
8	27.2	4.0	3.2
9	30.6	4.5	3.6

From the top:

For **19°+** or **199°+**, read as printed; for **109°+** or **289°+**, read co-function.

From the bottom:

For **70°+** or **250°+**, read as printed; for **160°+** or **340°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.53 405	35	9.56 107	39	10.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	10.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	10.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	10.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	10.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	10.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	10.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	10.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	10.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	10.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	10.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	10.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	10.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	10.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	10.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	10.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	10.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	10.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	10.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	10.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	10.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	10.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	10.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	10.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	10.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	10.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	10.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	10.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	10.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	10.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	10.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	10.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	10.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	10.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	10.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	10.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	10.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	10.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	10.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	10.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	10.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	10.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	10.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	10.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	10.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	10.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	10.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	10.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	10.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	10.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	10.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	10.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	10.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	10.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	10.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	10.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	10.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	10.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	10.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	10.41 620	9.97 020	5	1
60	9.55 433		9.58 418		10.41 582	9.97 015		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	40	39	38
2	8.0	7.8	7.6
3	12.0	11.7	11.4
4	16.0	15.6	15.2
5	20.0	19.5	19.0
6	24.0	23.4	22.8
7	28.0	27.3	26.6
8	32.0	31.2	30.4
9	36.0	35.1	34.2

	37	35	34
2	7.4	7.0	6.8
3	11.1	10.5	10.2
4	14.8	14.0	13.6
5	18.5	17.5	17.0
6	22.2	21.0	20.4
7	25.9	24.5	23.8
8	29.6	28.0	27.2
9	33.3	31.5	30.6

	33	5	4
2	6.6	1.0	0.8
3	9.9	1.5	1.2
4	13.2	2.0	1.6
5	16.5	2.5	2.0
6	19.8	3.0	2.4
7	23.1	3.5	2.8
8	26.4	4.0	3.2
9	29.7	4.5	3.6

From the top:

For 20°+ or 200°+, read as printed; for 110°+ or 290°+, read co-function.

From the bottom:

For 69°+ or 249°+, read as printed; for 159°+ or 339°+, read co-function.

Prop. Pts.

69° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.55 433	33	9.58 418	37	10.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	10.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	10.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	10.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	10.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	10.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	10.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	10.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	10.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	10.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	10.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	10.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	10.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	10.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	10.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	10.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	10.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	10.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	10.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	10.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	10.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	10.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	10.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	10.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	10.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	10.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	10.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	10.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	10.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	10.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	10.40 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	10.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	10.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	10.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	10.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	10.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	10.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	10.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	10.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	10.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	10.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	10.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	10.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	10.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	10.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	10.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	10.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	10.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	10.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	10.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	10.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	10.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	10.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	10.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	10.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	10.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	10.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	10.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	10.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	10.39 395	9.96 722	5	1
60	9.57 358		9.60 641		10.39 359	9.96 717		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	38	37	36
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	33	32	31
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	6	5	4
2	1.2	1.0	0.8
3	1.8	1.5	1.2
4	2.4	2.0	1.6
5	3.0	2.5	2.0
6	3.6	3.0	2.4
7	4.2	3.5	2.8
8	4.8	4.0	3.2
9	5.4	4.5	3.6

From the top:

For **21°+** or **201°+**, read as printed; for **111°+** or **291°+**, read co-function.

From the bottom:

For **68°+** or **248°+**, read as printed; for **158°+** or **338°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.57 358		9.60 641		10.39 359	9.96 717		**60**
1	9.57 389	31	9.60 677	36	10.39 323	9.96 711	6	59
2	9.57 420	31	9.60 714	37	10.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	10.39 250	9.96 701	5	57
4	9.57 482	31	9.60 786	36	10.39 214	9.96 696	5	56
5	9.57 514	32	9.60 823	37	10.39 177	9.96 691	5	**55**
6	9.57 545	31	9.60 859	36	10.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	10.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	10.39 069	9.96 676	5	52
9	9.57 638	31	9.60 967	36	10.39 033	9.96 670	6	51
10	9.57 669	31	9.61 004	37	10.38 996	9.96 665	5	**50**
11	9.57 700	31	9.61 040	36	10.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	10.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	10.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	10.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	10.38 816	9.96 640	5	**45**
16	9.57 855	31	9.61 220	36	10.38 780	9.96 634	6	44
17	9.57 885	30	9.61 256	36	10.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	10.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	10.38 672	9.96 619	5	41
20	9.57 978	31	9.61 364	36	10.38 636	9.96 614	5	**40**
21	9.58 008	30	9.61 400	36	10.38 600	9.96 608	6	39
22	9.58 039	31	9.61 436	36	10.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	10.38 528	9.96 598	5	37
24	9.58 101	31	9.61 508	36	10.38 492	9.96 593	5	36
25	9.58 131	30	9.61 544	36	10.38 456	9.96 588	5	**35**
26	9.58 162	31	9.61 579	35	10.38 421	9.96 582	6	34
27	9.58 192	30	9.61 615	36	10.38 385	9.96 577	5	33
28	9.58 223	31	9.61 651	36	10.38 349	9.96 572	5	32
29	9.58 253	30	9.61 687	36	10.38 313	9.96 567	5	31
30	9.58 284	31	9.61 722	35	10.38 278	9.96 562	5	**30**
31	9.58 314	30	9.61 758	36	10.38 242	9.96 556	6	29
32	9.58 345	31	9.61 794	36	10.38 206	9.96 551	5	28
33	9.58 375	30	9.61 830	36	10.38 170	9.96 546	5	27
34	9.58 406	31	9.61 865	35	10.38 135	9.96 541	5	26
35	9.58 436	30	9.61 901	36	10.38 099	9.96 535	6	**25**
36	9.58 467	31	9.61 936	35	10.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	10.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	36	10.37 992	9.96 520	5	22
39	9.58 557	30	9.62 043	35	10.37 957	9.96 514	6	21
40	9.58 588	31	9.62 079	36	10.37 921	9.96 509	5	**20**
41	9.58 618	30	9.62 114	35	10.37 886	9.96 504	5	19
42	9.58 648	30	9.62 150	36	10.37 850	9.96 498	6	18
43	9.58 678	30	9.62 185	35	10.37 815	9.96 493	5	17
44	9.58 709	31	9.62 221	36	10.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	35	10.37 744	9.96 483	5	**15**
46	9.58 769	30	9.62 292	36	10.37 708	9.96 477	6	14
47	9.58 799	30	9.62 327	35	10.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	35	10.37 638	9.96 467	5	12
49	9.58 859	30	9.62 398	36	10.37 602	9.96 461	6	11
50	9.58 889	30	9.62 433	35	10.37 567	9.96 456	5	**10**
51	9.58 919	30	9.62 468	35	10.37 532	9.96 451	5	9
52	9.58 949	30	9.62 504	36	10.37 496	9.96 445	6	8
53	9.58 979	30	9.62 539	35	10.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	10.37 426	9.96 435	5	6
55	9.59 039	30	9.62 609	35	10.37 391	9.96 429	6	**5**
56	9.59 069	30	9.62 645	36	10.37 355	9.96 424	5	4
57	9.59 098	29	9.62 680	35	10.37 320	9.96 419	5	3
58	9.59 128	30	9.62 715	35	10.37 285	9.96 413	6	2
59	9.59 158	30	9.62 750	35	10.37 250	9.96 408	5	1
60	9.59 188	30	9.62 785	35	10.37 215	9.96 403	5	**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	37	36	35
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	32	31	30
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

	29	6	5
2	5.8	1.2	1.0
3	8.7	1.8	1.5
4	11.6	2.4	2.0
5	14.5	3.0	2.5
6	17.4	3.6	3.0
7	20.3	4.2	3.5
8	23.2	4.8	4.0
9	26.1	5.4	4.5

From the top:

For **22°+** or **202°+**, read as printed; for **112°+** or **292°+**, read co-function.

From the bottom:

For **67°+** or **247°+**, read as printed; for **157°+** or **337°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.59 188	30	9.62 785	35	10.37 215	9.96 403	6	60
1	9.59 218	29	9.62 820	35	10.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	10.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	10.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	10.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	10.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	10.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	10.36 969	9.96 365	5	53
8	9.59 425	30	9.63 066	35	10.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	10.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	10.36 865	9.96 349	6	50
11	9.59 514	29	9.63 170	35	10.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	10.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	10.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	10.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	10.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	10.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	10.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	10.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	10.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	10.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	10.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	10.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	10.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	10.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	10.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	10.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	10.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	10.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	10.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	10.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865	34	10.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	10.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	10.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	10.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	10.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	10.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	10.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	10.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	10.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	10.35 825	9.96 185	6	20
41	9.60 388	29	9.64 209	34	10.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	10.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	10.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	10.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	10.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	10.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	10.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	10.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	10.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	10.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	10.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	10.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	10.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	10.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	10.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	10.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	10.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	10.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	10.35 176	9.96 079	6	1
60	9.60 931		9.64 858		10.35 142	9.96 073		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	36	35
2	7.2	7.0
3	10.8	10.5
4	14.4	14.0
5	18.0	17.5
6	21.6	21.0
7	25.2	24.5
8	28.8	28.0
9	32.4	31.5

	34	30
2	6.8	6.0
3	10.2	9.0
4	13.6	12.0
5	17.0	15.0
6	20.4	18.0
7	23.8	21.0
8	27.2	24.0
9	30.6	27.0

	29	28
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	6	5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

From the top:

For **23°+** or **203°+**, read as printed; for **113°+** or **293°+**, read co-function.

From the bottom:

For **66°+** or **246°+**, read as printed; for **156°+** or **336°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.60 931		9.64 858		10.35 142	9.96 073		**60**
1	9.60 960	29	9.64 892	34	10.35 108	9.96 067	6	59
2	9.60 988	28	9.64 926	34	10.35 074	9.96 062	5	58
3	9.61 016	28	9.64 960	34	10.35 040	9.96 056	6	57
4	9.61 045	29	9.64 994	34	10.35 006	9.96 050	6	56
5	9.61 073	28	9.65 028	34	10.34 972	9.96 045	5	**55**
6	9.61 101	28	9.65 062	34	10.34 938	9.96 039	6	54
7	9.61 129	28	9.65 096	34	10.34 904	9.96 034	5	53
8	9.61 158	29	9.65 130	34	10.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	34	10.34 836	9.96 022	6	51
10	9.61 214	28	9.65 197	33	10.34 803	9.96 017	5	**50**
11	9.61 242	28	9.65 231	34	10.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	10.34 735	9.96 005	6	48
13	9.61 298	28	9.65 299	34	10.34 701	9.96 000	5	47
14	9.61 326	28	9.65 333	34	10.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	33	10.34 634	9.95 988	6	**45**
16	9.61 382	28	9.65 400	34	10.34 600	9.95 982	6	44
17	9.61 411	29	9.65 434	34	10.34 566	9.95 977	5	43
18	9.61 438	27	9.65 467	33	10.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	10.34 499	9.95 965	6	41
20	9.61 494	28	9.65 535	34	10.34 465	9.95 960	5	**40**
21	9.61 522	28	9.65 568	33	10.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	10.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	34	10.34 364	9.95 942	6	37
24	9.61 606	28	9.65 669	33	10.34 331	9.95 937	5	36
25	9.61 634	28	9.65 703	34	10.34 297	9.95 931	6	**35**
26	9.61 662	28	9.65 736	33	10.34 264	9.95 925	6	34
27	9.61 689	27	9.65 770	34	10.34 230	9.95 920	5	33
28	9.61 717	28	9.65 803	33	10.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	34	10.34 163	9.95 908	6	31
30	9.61 773	28	9.65 870	33	10.34 130	9.95 902	6	**30**
31	9.61 800	27	9.65 904	34	10.34 096	9.95 897	5	29
32	9.61 828	28	9.65 937	33	10.34 063	9.95 891	6	28
33	9.61 856	28	9.65 971	34	10.34 029	9.95 885	6	27
34	9.61 883	27	9.66 004	33	10.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	34	10.33 962	9.95 873	6	**25**
36	9.61 939	28	9.66 071	33	10.33 929	9.95 868	5	24
37	9.61 966	27	9.66 104	33	10.33 896	9.95 862	6	23
38	9.61 994	28	9.66 138	34	10.33 862	9.95 856	6	22
39	9.62 021	27	9.66 171	33	10.33 829	9.95 850	6	21
40	9.62 049	28	9.66 204	33	10.33 796	9.95 844	6	**20**
41	9.62 076	27	9.66 238	34	10.33 762	9.95 839	5	19
42	9.62 104	28	9.66 271	33	10.33 729	9.95 833	6	18
43	9.62 131	27	9.66 304	33	10.33 696	9.95 827	6	17
44	9.62 159	28	9.66 337	33	10.33 663	9.95 821	6	16
45	9.62 186	27	9.66 371	34	10.33 629	9.95 815	6	**15**
46	9.62 214	28	9.66 404	33	10.33 596	9.95 810	5	14
47	9.62 241	27	9.66 437	33	10.33 563	9.95 804	6	13
48	9.62 268	27	9.66 470	33	10.33 530	9.95 798	6	12
49	9.62 296	28	9.66 503	33	10.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	34	10.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	10.33 430	9.95 780	6	9
52	9.62 377	27	9.66 603	33	10.33 397	9.95 775	5	8
53	9.62 405	28	9.66 636	33	10.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	10.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	10.33 298	9.95 757	6	**5**
56	9.62 486	27	9.66 735	33	10.33 265	9.95 751	6	4
57	9.62 513	27	9.66 768	33	10.33 232	9.95 745	6	3
58	9.62 541	28	9.66 801	33	10.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	10.33 166	9.95 733	6	1
60	9.62 595	27	9.66 867	33	10.33 133	9.95 728	5	**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	34	33
2	6.8	6.6
3	10.2	9.9
4	13.6	13.2
5	17.0	16.5
6	20.4	19.8
7	23.8	23.1
8	27.2	26.4
9	30.6	29.7

	29	28
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	6
2	5.4	1.2
3	8.1	1.8
4	10.8	2.4
5	13.5	3.0
6	16.2	3.6
7	18.9	4.2
8	21.6	4.8
9	24.3	5.4

	5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

From the top:

For **24°+** or **204°+**, read as printed; for **114°+** or **294°+**, read co-function.

From the bottom:

For **65°+** or **245°+**, read as printed; for **155°+** or **335°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.62 595	27	9.66 867	33	10.33 133	9.95 728	6	**60**
1	9.62 622	27	9.66 900	33	10.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	10.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	10.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	10.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	10.32 968	9.95 698	6	**55**
6	9.62 757	27	9.67 065	33	10.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	10.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	10.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	10.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	10.32 804	9.95 668	5	**50**
11	9.62 892	26	9.67 229	33	10.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	10.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	10.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	10.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	10.32 640	9.95 639	6	**45**
16	9.63 026	26	9.67 393	33	10.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	10.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	10.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	10.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	10.32 476	9.95 609	6	**40**
21	9.63 159	27	9.67 556	33	10.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	10.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	10.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	10.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	10.32 313	9.95 579	6	**35**
26	9.63 292	27	9.67 719	33	10.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	10.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	10.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	10.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	10.32 150	9.95 549	6	**30**
31	9.63 425	26	9.67 882	33	10.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	10.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	10.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	10.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	10.31 988	9.95 519	6	**25**
36	9.63 557	26	9.68 044	33	10.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	10.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	10.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	10.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	10.31 826	9.95 488	6	**20**
41	9.63 689	26	9.68 206	33	10.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	10.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	10.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	10.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	10.31 664	9.95 458	6	**15**
46	9.63 820	26	9.68 368	32	10.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	10.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	10.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	10.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	10.31 503	9.95 427	6	**10**
51	9.63 950	26	9.68 529	32	10.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	10.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	10.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	10.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	10.31 342	9.95 397	6	**5**
56	9.64 080	26	9.68 690	32	10.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	10.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	10.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	10.31 214	9.95 372	6	1
60	9.64 184		9.68 818		10.31 182	9.95 366		**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	33	32
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	27	26
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	7	6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

From the top:

For **25°+** or **205°+**, read as printed; for **115°+** or **295°+**, read co-function.

From the bottom:

For **64°+** or **244°+**, read as printed; for **154°+** or **334°+**, read co-function.

Prop. Pts.

64° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.64 184	26	9.68 818	32	10.31 182	9.95 366	6	**60**
1	9.64 210	26	9.68 850	32	10.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	10.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	10.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	10.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	10.31 022	9.95 335	6	**55**
6	9.64 339	26	9.69 010	32	10.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	10.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	10.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	10.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	10.30 862	9.95 304	6	**50**
11	9.64 468	26	9.69 170	32	10.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	10.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	10.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	10.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	10.30 702	9.95 273	6	**45**
16	9.64 596	26	9.69 329	32	10.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	10.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	10.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	10.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	10.30 543	9.95 242	6	**40**
21	9.64 724	25	9.69 488	32	10.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	10.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	10.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	10.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	10.30 385	9.95 211	7	**35**
26	9.64 851	26	9.69 647	32	10.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	10.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	10.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	10.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	10.30 226	9.95 179	6	**30**
31	9.64 978	25	9.69 805	32	10.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	10.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	10.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	10.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	10.30 068	9.95 148	7	**25**
36	9.65 104	26	9.69 963	32	10.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	10.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	10.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	10.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	10.29 911	9.95 116	6	**20**
41	9.65 230	25	9.70 121	31	10.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	10.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	10.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	10.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	10.29 753	9.95 084	6	**15**
46	9.65 356	25	9.70 278	31	10.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	10.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	10.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	10.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	10.29 596	9.95 052	6	**10**
51	9.65 481	25	9.70 435	31	10.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	10.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	10.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	10.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	10.29 440	9.95 020	6	**5**
56	9.65 605	25	9.70 592	31	10.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	10.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	10.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	10.29 315	9.94 995	7	1
60	9.65 705		9.70 717		10.29 283	9.94 988		**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	32	31
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	24	7
2	4.8	1.4
3	7.2	2.1
4	9.6	2.8
5	12.0	3.5
6	14.4	4.2
7	16.8	4.9
8	19.2	5.6
9	21.6	6.3

	6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

From the top:

For **26°+** or **206°+**, read as printed; for **116°+** or **296°+**, read co-function.

From the bottom:

For **63°+** or **243°+**, read as printed; for **153°+** or **333°+**, read co-function.

Prop. Pts.

'	L Sin	d	L Tan	c d	L Ctn	L Cos	d	'
0	9.65 705		9.70 717		10.29 283	9.94 988		60
1	9.65 729	24	9.70 748	31	10.29 252	9.94 982	6	59
2	9.65 754	25	9.70 779	31	10.29 221	9.94 975	7	58
3	9.65 779	25	9.70 810	31	10.29 190	9.94 969	6	57
4	9.65 804	25	9.70 841	31	10.29 159	9.94 962	7	56
5	9.65 828	24	9.70 873	32	10.29 127	9.94 956	6	55
6	9.65 853	25	9.70 904	31	10.29 096	9.94 949	7	54
7	9.65 878	25	9.70 935	31	10.29 065	9.94 943	6	53
8	9.65 902	24	9.70 966	31	10.29 034	9.94 936	7	52
9	9.65 927	25	9.70 997	31	10.29 003	9.94 930	6	51
10	9.65 952	25	9.71 028	31	10.28 972	9.94 923	7	50
11	9.65 976	24	9.71 059	31	10.28 941	9.94 917	6	49
12	9.66 001	25	9.71 090	31	10.28 910	9.94 911	6	48
13	9.66 025	24	9.71 121	31	10.28 879	9.94 904	7	47
14	9.66 050	25	9.71 153	32	10.28 847	9.94 898	6	46
15	9.66 075	25	9.71 184	31	10.28 816	9.94 891	7	45
16	9.66 099	24	9.71 215	31	10.28 785	9.94 885	6	44
17	9.66 124	25	9.71 246	31	10.28 754	9.94 878	7	43
18	9.66 148	24	9.71 277	31	10.28 723	9.94 871	7	42
19	9.66 173	25	9.71 308	31	10.28 692	9.94 865	6	41
20	9.66 197	24	9.71 339	31	10.28 661	9.94 858	7	40
21	9.66 221	24	9.71 370	31	10.28 630	9.94 852	6	39
22	9.66 246	25	9.71 401	31	10.28 599	9.94 845	7	38
23	9.66 270	24	9.71 431	30	10.28 569	9.94 839	6	37
24	9.66 295	25	9.71 462	31	10.28 538	9.94 832	7	36
25	9.66 319	24	9.71 493	31	10.28 507	9.94 826	6	35
26	9.66 343	24	9.71 524	31	10.28 476	9.94 819	7	34
27	9.66 368	25	9.71 555	31	10.28 445	9.94 813	6	33
28	9.66 392	24	9.71 586	31	10.28 414	9.94 806	7	32
29	9.66 416	24	9.71 617	31	10.28 383	9.94 799	7	31
30	9.66 441	25	9.71 648	31	10.28 352	9.94 793	6	30
31	9.66 465	24	9.71 679	31	10.28 321	9.94 786	7	29
32	9.66 489	24	9.71 709	30	10.28 291	9.94 780	6	28
33	9.66 513	24	9.71 740	31	10.28 260	9.94 773	7	27
34	9.66 537	24	9.71 771	31	10.28 229	9.94 767	6	26
35	9.66 562	25	9.71 802	31	10.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	31	10.28 167	9.94 753	7	24
37	9.66 610	24	9.71 863	30	10.28 137	9.94 747	6	23
38	9.66 634	24	9.71 894	31	10.28 106	9.94 740	7	22
39	9.66 658	24	9.71 925	31	10.28 075	9.94 734	6	21
40	9.66 682	24	9.71 955	30	10.28 045	9.94 727	7	20
41	9.66 706	24	9.71 986	31	10.28 014	9.94 720	7	19
42	9.66 731	25	9.72 017	31	10.27 983	9.94 714	6	18
43	9.66 755	24	9.72 048	31	10.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	30	10.27 922	9.94 700	7	16
45	9.66 803	24	9.72 109	31	10.27 891	9.94 694	6	15
46	9.66 827	24	9.72 140	31	10.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	30	10.27 830	9.94 680	7	13
48	9.66 875	24	9.72 201	31	10.27 799	9.94 674	6	12
49	9.66 899	24	9.72 231	30	10.27 769	9.94 667	7	11
50	9.66 922	23	9.72 262	31	10.27 738	9.94 660	7	10
51	9.66 946	24	9.72 293	31	10.27 707	9.94 654	6	9
52	9.66 970	24	9.72 323	30	10.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	31	10.27 646	9.94 640	7	7
54	9.67 018	24	9.72 384	30	10.27 616	9.94 634	6	6
55	9.67 042	24	9.72 415	31	10.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	30	10.27 555	9.94 620	7	4
57	9.67 090	24	9.72 476	31	10.27 524	9.94 614	6	3
58	9.67 113	23	9.72 506	30	10.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	31	10.27 463	9.94 600	7	1
60	9.67 161	24	9.72 567	30	10.27 433	9.94 593	7	0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	'

Prop. Pts.

	32	31
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	30	25
2	6.0	5.0
3	9.0	7.5
4	12.0	10.0
5	15.0	12.5
6	18.0	15.0
7	21.0	17.5
8	24.0	20.0
9	27.0	22.5

	24	23
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	7	6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

From the top:

For **27°+** or **207°+**, read as printed; for **117°+** or **297°+**, read co-function.

From the bottom:

For **62°+** or **242°+**, read as printed; for **152°+** or **332°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.67 161	24	9.72 567	31	10.27 433	9.94 593	6	60
1	9.67 185	23	9.72 598	30	10.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	10.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	10.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	10.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	10.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	10.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	10.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	10.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	10.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	10.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	10.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	10.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	10.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	10.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	10.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	10.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	10.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	10.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	10.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	30	10.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205	30	10.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	10.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	10.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	10.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	10.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	10.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	10.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	10.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	10.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	10.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	10.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	10.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	10.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	10.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	10.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	10.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	10.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	10.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	10.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	10.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	10.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	10.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	10.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	10.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	10.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	10.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	10.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	10.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	10.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	10.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	10.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	10.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	10.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	10.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	10.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	10.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	10.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	10.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	10.25 655	9.94 189	7	1
60	9.68 557		9.74 375		10.25 625	9.94 182		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	31	30
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29	24
2	5.8	4.8
3	8.7	7.2
4	11.6	9.6
5	14.5	12.0
6	17.4	14.4
7	20.3	16.8
8	23.2	19.2
9	26.1	21.6

	23	22
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	7	6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

From the top:

For **28°+** or **208°+**, read as printed; for **118°+** or **298°+**, read co-function.

From the bottom:

For **61°+** or **241°+**, read as printed; for **151°+** or **331°+**, read co-function.

Prop. Pts.

61° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.68 557		9.74 375		10.25 625	9.94 182		**60**
1	9.68 580	23	9.74 405	30	10.25 595	9.94 175	7	59
2	9.68 603	23	9.74 435	30	10.25 565	9.94 168	7	58
3	9.68 625	22	9.74 465	30	10.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	29	10.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	10.25 476	9.94 147	7	**55**
6	9.68 694	23	9.74 554	30	10.25 446	9.94 140	7	54
7	9.68 716	22	9.74 583	29	10.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	10.25 387	9.94 126	7	52
9	9.68 762	23	9.74 643	30	10.25 357	9.94 119	7	51
10	9.68 784	22	9.74 673	30	10.25 327	9.94 112	7	**50**
11	9.68 807	23	9.74 702	29	10.25 298	9.94 105	7	49
12	9.68 829	22	9.74 732	30	10.25 268	9.94 098	7	48
13	9.68 852	23	9.74 762	30	10.25 238	9.94 090	8	47
14	9.68 875	23	9.74 791	29	10.25 209	9.94 083	7	46
15	9.68 897	22	9.74 821	30	10.25 179	9.94 076	7	**45**
16	9.68 920	23	9.74 851	30	10.25 149	9.94 069	7	44
17	9.68 942	22	9.74 880	29	10.25 120	9.94 062	7	43
18	9.68 965	23	9.74 910	30	10.25 090	9.94 055	7	42
19	9.68 987	22	9.74 939	29	10.25 061	9.94 048	7	41
20	9.69 010	23	9.74 969	30	10.25 031	9.94 041	7	**40**
21	9.69 032	22	9.74 998	29	10.25 002	9.94 034	7	39
22	9.69 055	23	9.75 028	30	10.24 972	9.94 027	7	38
23	9.69 077	22	9.75 058	30	10.24 942	9.94 020	7	37
24	9.69 100	23	9.75 087	29	10.24 913	9.94 012	8	36
25	9.69 122	22	9.75 117	30	10.24 883	9.94 005	7	**35**
26	9.69 144	22	9.75 146	29	10.24 854	9.93 998	7	34
27	9.69 167	23	9.75 176	30	10.24 824	9.93 991	7	33
28	9.69 189	22	9.75 205	29	10.24 795	9.93 984	7	32
29	9.69 212	23	9.75 235	30	10.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	29	10.24 736	9.93 970	7	**30**
31	9.69 256	22	9.75 294	30	10.24 706	9.93 963	7	29
32	9.69 279	23	9.75 323	29	10.24 677	9.93 955	8	28
33	9.69 301	22	9.75 353	30	10.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	10.24 618	9.93 941	7	26
35	9.69 345	22	9.75 411	29	10.24 589	9.93 934	7	**25**
36	9.69 368	23	9.75 441	30	10.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	29	10.24 530	9.93 920	7	23
38	9.69 412	22	9.75 500	30	10.24 500	9.93 912	8	22
39	9.69 434	22	9.75 529	29	10.24 471	9.93 905	7	21
40	9.69 456	22	9.75 558	29	10.24 442	9.93 898	7	**20**
41	9.69 479	23	9.75 588	30	10.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	29	10.24 383	9.93 884	7	18
43	9.69 523	22	9.75 647	30	10.24 353	9.93 876	8	17
44	9.69 545	22	9.75 676	29	10.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	29	10.24 295	9.93 862	7	**15**
46	9.69 589	22	9.75 735	30	10.24 265	9.93 855	7	14
47	9.69 611	22	9.75 764	29	10.24 236	9.93 847	8	13
48	9.69 633	22	9.75 793	29	10.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	29	10.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	30	10.24 148	9.93 826	7	**10**
51	9.69 699	22	9.75 881	29	10.24 119	9.93 819	7	9
52	9.69 721	22	9.75 910	29	10.24 090	9.93 811	8	8
53	9.69 743	22	9.75 939	29	10.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	30	10.24 031	9.93 797	7	6
55	9.69 787	22	9.75 998	29	10.24 002	9.93 789	8	**5**
56	9.69 809	22	9.76 027	29	10.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	29	10.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	30	10.23 914	9.93 768	7	2
59	9.69 875	22	9.76 115	29	10.23 885	9.93 760	8	1
60	9.69 897	22	9.76 144	29	10.23 856	9.93 753	7	**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	30	29
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	23	22
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	8	7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

From the top:

For 29°+ or 209°+, read as printed; for 119°+ or 299°+, read co-function.

From the bottom:

For 60°+ or 240°+, read as printed; for 150°+ or 330°+, read co-function.

Prop. Pts.

60° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.69 897	22	9.76 144	29	10.23 856	9.93 753	7	**60**
1	9.69 919	22	9.76 173	29	10.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	10.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	10.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	10.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	10.23 710	9.93 717	8	**55**
6	9.70 028	22	9.76 319	29	10.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	10.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	10.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	10.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	10.23 565	9.93 680	7	**50**
11	9.70 137	22	9.76 464	29	10.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	10.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	10.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	10.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	10.23 420	9.93 643	7	**45**
16	9.70 245	22	9.76 609	30	10.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	10.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	10.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	10.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	10.23 275	9.93 606	7	**40**
21	9.70 353	22	9.76 754	29	10.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	10.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	10.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	10.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	10.23 130	9.93 569	7	**35**
26	9.70 461	21	9.76 899	29	10.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	10.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	10.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	10.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	10.22 985	9.93 532	7	**30**
31	9.70 568	22	9.77 044	29	10.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	10.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	10.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	10.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	10.22 841	9.93 495	8	**25**
36	9.70 675	22	9.77 188	29	10.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	10.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	10.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	10.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	10.22 697	9.93 457	7	**20**
41	9.70 782	21	9.77 332	29	10.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	10.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	10.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	10.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	10.22 553	9.93 420	8	**15**
46	9.70 888	21	9.77 476	29	10.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	10.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	10.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	10.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	10.22 409	9.93 382	7	**10**
51	9.70 994	21	9.77 619	29	10.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	10.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	10.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	10.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	10.22 266	9.93 344	7	**5**
56	9.71 100	21	9.77 763	28	10.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	10.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	10.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	10.22 151	9.93 314	7	1
60	9.71 184		9.77 877		10.22 123	9.93 307		**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	30	29
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	28	22
2	5.6	4.4
3	8.4	6.6
4	11.2	8.8
5	14.0	11.0
6	16.8	13.2
7	19.6	15.4
8	22.4	17.6
9	25.2	19.8

	21	8
2	4.2	1.6
3	6.3	2.4
4	8.4	3.2
5	10.5	4.0
6	12.6	4.8
7	14.7	5.6
8	16.8	6.4
9	18.9	7.2

	7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

From the top:
For 30°+ or 210°+, read as printed; for 120°+ or 300°+, read co-function.

From the bottom:
For 59°+ or 239°+, read as printed; for 149°+ or 329°+, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.71 184	21	9.77 877	29	10.22 123	9.93 307	8	60
1	9.71 205	21	9.77 906	29	10.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	10.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	10.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	10.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	10.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	10.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	10.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	10.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	10.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	10.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	10.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	10.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	10.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	10.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	10.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	10.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	10.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	10.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	10.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	10.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	10.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	10.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	10.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	10.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	10.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	10.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	10.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	10.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	10.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	10.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	10.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	10.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	10.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	10.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	10.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	10.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	10.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	10.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	10.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	10.20 985	9.92 999	8	20
41	9.72 034	21	9.79 043	29	10.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	10.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	10.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	10.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	10.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	10.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	10.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	10.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	10.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	10.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	10.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	10.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	10.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	10.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	10.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	10.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	10.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	10.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	10.20 449	9.92 850	8	1
60	9.72 421		9.79 579		10.20 421	9.92 842		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	29	28
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	21	20
2	4.2	4.0
3	6.3	6.0
4	8.4	8.0
5	10.5	10.0
6	12.6	12.0
7	14.7	14.0
8	16.8	16.0
9	18.9	18.0

	8	7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

From the top: For **31°+** or **211°+**, read as printed; for **121°+** or **301°+**, read co-function.

From the bottom: For **58°+** or **238°+**, read as printed; for **148°+** or **328°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.72 421	20	9.79 579	28	10.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	10.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	10.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	10.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	10.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	10.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	10.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	10.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	10.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	10.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	10.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	10.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	10.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	10.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	10.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	10.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	10.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	10.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	10.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	10.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	10.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	10.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	10.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	10.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	10.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	10.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	10.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	10.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	10.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	10.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	10.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	10.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	10.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	10.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	10.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	10.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	10.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	10.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	10.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	10.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	10.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	10.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	10.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	10.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	10.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	10.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	10.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	10.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	10.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	10.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	10.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	10.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	10.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	10.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	10.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	10.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	10.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	10.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	10.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	10.18 776	9.92 367	8	1
60	9.73 611		9.81 252		10.18 748	9.92 359		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	29	28
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	21
2	5.4	4.2
3	8.1	6.3
4	10.8	8.4
5	13.5	10.5
6	16.2	12.6
7	18.9	14.7
8	21.6	16.8
9	24.3	18.9

	20	19
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	9	8	7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

From the top:

For **32°+** or **212°+**, read as printed; for **122°+** or **302°+**, read co-function.

From the bottom:

For **57°+** or **237°+**, read as printed; for **147°+** or **327°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.73 611	19	9.81 252	27	10.18 748	9.92 359	8	**60**
1	9.73 630	20	9.81 279	28	10.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	10.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	10.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	10.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	10.18 610	9.92 318	8	**55**
6	9.73 727	20	9.81 418	27	10.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	10.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	10.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	10.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	10.18 472	9.92 277	8	**50**
11	9.73 824	19	9.81 556	27	10.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	10.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	10.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	10.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	10.18 334	9.92 235	8	**45**
16	9.73 921	19	9.81 693	28	10.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	10.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	10.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	10.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	10.18 197	9.92 194	8	**40**
21	9.74 017	19	9.81 831	27	10.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	10.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	10.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	10.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	10.18 059	9.92 152	8	**35**
26	9.74 113	19	9.81 968	28	10.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	10.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	10.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	10.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	10.17 922	9.92 111	9	**30**
31	9.74 208	19	9.82 106	27	10.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	10.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	10.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	10.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	10.17 785	9.92 069	9	**25**
36	9.74 303	19	9.82 243	27	10.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	10.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	10.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	10.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	10.17 648	9.92 027	9	**20**
41	9.74 398	19	9.82 380	27	10.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	10.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	10.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	10.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	10.17 511	9.91 985	9	**15**
46	9.74 493	19	9.82 517	27	10.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	10.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	10.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	10.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	10.17 374	9.91 942	8	**10**
51	9.74 587	19	9.82 653	28	10.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	10.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	10.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	10.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	10.17 238	9.91 900	9	**5**
56	9.74 681	19	9.82 790	27	10.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	10.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	10.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	10.17 129	9.91 866	9	1
60	9.74 756		9.82 899		10.17 101	9.91 857		**0**
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	28	27
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	20	19
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1

	18	9
2	3.6	1.8
3	5.4	2.7
4	7.2	3.6
5	9.0	4.5
6	10.8	5.4
7	12.6	6.3
8	14.4	7.2
9	16.2	8.1

	8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

From the top:
For **33°+** or **213°+**, read as printed; for **123°+** or **303°+**, read co-function.

From the bottom:
For **56°+** or **236°+**, read as printed; for **146°+** or **326°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	c d	L Ctn	L Cos	d	
0	9.74 756	19	9.82 899	27	10.17 101	9.91 857	8	60
1	9.74 775	19	9.82 926	27	10.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	10.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	10.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	10.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	10.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	10.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	10.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	10.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	10.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	10.16 829	9.91 772	9	50
11	9.74 961	19	9.83 198	27	10.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	10.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	10.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	10.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	10.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	10.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	10.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	10.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	10.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	10.16 558	9.91 686	9	40
21	9.75 147	18	9.83 470	27	10.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	10.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	10.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	10.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	10.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	10.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	10.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	10.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	10.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	10.16 287	9.91 599	8	30
31	9.75 331	19	9.83 740	28	10.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	10.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	10.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	10.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	10.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	10.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	10.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	10.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	10.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	10.16 016	9.91 512	8	20
41	9.75 514	19	9.84 011	27	10.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	10.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	10.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	10.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	10.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	10.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	10.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	10.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	10.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	10.15 746	9.91 425	9	10
51	9.75 696	18	9.84 280	27	10.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	10.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	10.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	10.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	10.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	10.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	10.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	10.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	10.15 504	9.91 345	9	1
60	9.75 859		9.84 523		10.15 477	9.91 336		0
	L Cos	d	L Ctn	c d	L Tan	L Sin	d	′

Prop. Pts.

	28	27
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	26	19
2	5.2	3.8
3	7.8	5.7
4	10.4	7.6
5	13.0	9.5
6	15.6	11.4
7	18.2	13.3
8	20.8	15.2
9	23.4	17.1

	18	9
2	3.6	1.8
3	5.4	2.7
4	7.2	3.6
5	9.0	4.5
6	10.8	5.4
7	12.6	6.3
8	14.4	7.2
9	16.2	8.1

	8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

From the top:

For **34°+** or **214°+**, read as printed; for **124°+** or **304°+**, read co-function.

From the bottom:

For **55°+** or **235°+**, read as printed; for **145°+** or **325°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.75 859	18	9.84 523	27	10.15 477	9.91 336	8	**60**
1	9.75 877	18	9.84 550	26	10.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	10.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	10.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	10.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	10.15 343	9.91 292	9	**55**
6	9.75 967	18	9.84 684	27	10.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	10.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	10.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	10.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	10.15 209	9.91 248	9	**50**
11	9.76 057	18	9.84 818	27	10.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	10.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	10.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	10.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	10.15 075	9.91 203	9	**45**
16	9.76 146	18	9.84 952	27	10.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	10.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	10.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	10.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	10.14 941	9.91 158	9	**40**
21	9.76 236	17	9.85 086	27	10.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	10.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	10.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	10.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	10.14 807	9.91 114	9	**35**
26	9.76 324	18	9.85 220	27	10.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	10.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	10.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	10.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	10.14 673	9.91 069	9	**30**
31	9.76 413	18	9.85 354	26	10.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	10.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	10.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	10.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	10.14 540	9.91 023	9	**25**
36	9.76 501	18	9.85 487	27	10.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	10.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	10.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	10.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	10.14 406	9.90 978	9	**20**
41	9.76 590	17	9.85 620	27	10.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	10.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	10.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	10.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	10.14 273	9.90 933	9	**15**
46	9.76 677	18	9.85 754	26	10.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	10.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	10.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	10.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	10.14 140	9.90 887	9	**10**
51	9.76 765	17	9.85 887	26	10.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	10.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	10.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	10.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	10.14 007	9.90 842	10	**5**
56	9.76 852	18	9.86 020	26	10.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	10.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	10.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	10.13 900	9.90 805	9	1
60	9.76 922		9.86 126		10.13 874	9.90 796		**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	27	26
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

	10	9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

From the top:

For **35°+** or **215°+**, read as printed; for **125°+** or **305°+**, read co-function.

From the bottom:

For **54°+** or **234°+**, read as printed; for **144°+** or **324°+**, read co-function.

Prop. Pts.

54° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.76 922	17	9.86 126	27	10.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	10.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	10.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	10.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	10.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	10.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	10.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	10.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	10.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	10.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	10.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	10.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	10.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	10.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	10.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	10.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	10.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	10.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	10.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	10.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	10.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	10.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	10.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	10.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	10.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	10.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	10.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	10.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	10.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	10.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	10.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	10.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	10.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	10.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	10.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	10.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	10.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	10.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	10.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	10.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	10.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	10.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	10.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	10.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	10.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	10.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	10.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	10.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	10.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	10.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	10.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	10.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	10.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	10.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	10.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	10.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	10.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	10.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	10.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	10.12 315	9.90 244	9	1
60	9.77 946		9.87 711		10.12 289	9.90 235		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	27	26
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

	16	10
2	3.2	2.0
3	4.8	3.0
4	6.4	4.0
5	8.0	5.0
6	9.6	6.0
7	11.2	7.0
8	12.8	8.0
9	14.4	9.0

	9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

From the top:

For **36°+** or **216°+**, read as printed; for **126°+** or **306°+**, read co-function.

From the bottom:

For **53°+** or **233°+**, read as printed; for **143°+** or **323°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.77 946	17	9.87 711	27	10.12 289	9.90 235	10	60
1	9.77 963	17	9.87 738	26	10.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	10.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	10.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	10.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	10.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	10.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	10.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	10.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	10.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	10.12 026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	10.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	10.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	10.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	10.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	10.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	10.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	10.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	10.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	10.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	10.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	10.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	10.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	10.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	10.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	10.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	10.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	10.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	10.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	10.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	10.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	10.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	10.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	10.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	10.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	10.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	10.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	10.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	10.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	10.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	10.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	10.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	10.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	10.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	10.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	10.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	10.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	10.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	10.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	10.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	10.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	10.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	10.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	10.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	10.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	10.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	10.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	10.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	10.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	10.10 745	9.89 663	10	1
60	9.78 934		9.89 281		10.10 719	9.89 653		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	27	26
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	17	16
2	3.4	3.2
3	5.1	4.8
4	6.8	6.4
5	8.5	8.0
6	10.2	9.6
7	11.9	11.2
8	13.6	12.8
9	15.3	14.4

	10	9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

From the top:

For 37°+ or 217°+, read as printed; for 127°+ or 307°+, read co-function.

From the bottom:

For 52°+ or 232°+, read as printed; for 142°+ or 322°+, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.78 934	16	9.89 281	26	10.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	10.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	10.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	10.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	10.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	10.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	10.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	10.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	10.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	10.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	10.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	10.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	10.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	10.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	10.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	10.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	10.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	10.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	10.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	10.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	10.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	10.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	10.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	10.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	10.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	10.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	10.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	10.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	10.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	10.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	10.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	10.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	10.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	10.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	10.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	10.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	10.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	10.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	10.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	10.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	10.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	10.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	10.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	10.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	10.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	10.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	10.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	10.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	10.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	10.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	10.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	10.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	10.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	10.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	10.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	10.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	10.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	10.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	10.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	10.09 189	9.89 060	10	1
60	9.79 887		9.90 837		10.09 163	9.89 050		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	16
2	3.4	3.2
3	5.1	4.8
4	6.8	6.4
5	8.5	8.0
6	10.2	9.6
7	11.9	11.2
8	13.6	12.8
9	15.3	14.4

	15	11
2	3.0	2.2
3	4.5	3.3
4	6.0	4.4
5	7.5	5.5
6	9.0	6.6
7	10.5	7.7
8	12.0	8.8
9	13.5	9.9

	10	9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

From the top:

For 38°+ or 218°+, read as printed; for 128°+ or 308°+, read co-function.

From the bottom:

For 51°+ or 231°+, read as printed; for 141°+ or 321°+, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.79 887	16	9.90 837	26	10.09 163	9.89 050	10	**60**
1	9.79 903	15	9.90 863	26	10.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	10.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	10.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	10.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	10.09 034	9.88 999	10	**55**
6	9.79 981	15	9.90 992	26	10.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	10.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	10.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	10.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	10.08 905	9.88 948	11	**50**
11	9.80 058	16	9.91 121	26	10.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	10.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	10.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	10.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	10.08 776	9.88 896	10	**45**
16	9.80 136	15	9.91 250	26	10.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	10.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	10.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	10.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	10.08 647	9.88 844	10	**40**
21	9.80 213	15	9.91 379	25	10.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	10.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	10.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	10.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	10.08 518	9.88 793	11	**35**
26	9.80 290	15	9.91 507	26	10.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	10.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	10.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	10.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	10.08 390	9.88 741	11	**30**
31	9.80 366	16	9.91 636	26	10.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	10.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	10.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	10.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	10.08 261	9.88 688	10	**25**
36	9.80 443	15	9.91 765	26	10.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	10.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	10.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	10.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	10.08 132	9.88 636	10	**20**
41	9.80 519	15	9.91 893	26	10.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	10.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	10.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	10.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	10.08 004	9.88 584	11	**15**
46	9.80 595	15	9.92 022	26	10.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	10.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	10.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	10.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	10.07 875	9.88 531	10	**10**
51	9.80 671	15	9.92 150	26	10.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	10.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	10.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	10.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	10.07 747	9.88 478	10	**5**
56	9.80 746	16	9.92 279	25	10.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	10.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	10.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	10.07 644	9.88 436	11	1
60	9.80 807		9.92 381		10.07 619	9.88 425		**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	16	15
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

	11	10
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

From the top:

For **39°+** or **219°+**, read as printed; for **129°+** or **309°+**, read co-function.

From the bottom:

For **50°+** or **230°+**, read as printed; for **140°+** or **320°+**, read co-function.

Prop. Pts.

50° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.80 807	15	9.92 381	26	10.07 619	9.88 425	10	**60**
1	9.80 822	15	9.92 407	26	10.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	10.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	10.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	10.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	10.07 490	9.88 372	10	**55**
6	9.80 897	15	9.92 535	26	10.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	10.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	10.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	10.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	10.07 362	9.88 319	11	**50**
11	9.80 972	15	9.92 663	26	10.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	10.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	10.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	10.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	10.07 234	9.88 266	11	**45**
16	9.81 047	14	9.92 792	25	10.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	10.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	10.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	10.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	10.07 106	9.88 212	11	**40**
21	9.81 121	15	9.92 920	25	10.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	10.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	10.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	10.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	10.06 978	9.88 158	10	**35**
26	9.81 195	15	9.93 048	25	10.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	10.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	10.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	10.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	10.06 850	9.88 105	11	**30**
31	9.81 269	15	9.93 175	26	10.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	10.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	10.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	10.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	10.06 722	9.88 051	11	**25**
36	9.81 343	15	9.93 303	26	10.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	10.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	10.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	10.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	10.06 594	9.87 996	11	**20**
41	9.81 417	14	9.93 431	26	10.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	10.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	10.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	10.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	10.06 467	9.87 942	11	**15**
46	9.81 490	15	9.93 559	25	10.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	10.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	10.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	10.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	10.06 339	9.87 887	10	**10**
51	9.81 563	15	9.93 687	25	10.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	10.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	10.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	10.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	10.06 211	9.87 833	11	**5**
56	9.81 636	15	9.93 814	26	10.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	10.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	10.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	10.06 109	9.87 789	11	1
60	9.81 694		9.93 916		10.06 084	9.87 778		**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	11	10
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

From the top:
For **40°+** or **220°+**, read as printed; for **130°+** or **310°+**, read co-function.

From the bottom:
For **49°+** or **229°+**, read as printed; for **139°+** or **319°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.81 694	15	9.93 916	26	10.06 084	9.87 778	11	**60**
1	9.81 709	14	9.93 942	25	10.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	10.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	10.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	10.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	10.05 956	9.87 723	11	**55**
6	9.81 781	15	9.94 069	26	10.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	10.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	10.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	10.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	10.05 829	9.87 668	11	**50**
11	9.81 854	14	9.94 197	25	10.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	10.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	10.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	10.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	10.05 701	9.87 613	12	**45**
16	9.81 926	14	9.94 324	26	10.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	10.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	10.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	10.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	10.05 574	9.87 557	11	**40**
21	9.81 998	14	9.94 452	25	10.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	10.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	10.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	10.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	10.05 446	9.87 501	11	**35**
26	9.82 069	15	9.94 579	25	10.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	10.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	10.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	10.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	10.05 319	9.87 446	12	**30**
31	9.82 141	14	9.94 706	26	10.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	10.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	10.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	10.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	10.05 192	9.87 390	12	**25**
36	9.82 212	14	9.94 834	25	10.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	10.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	10.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	10.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	10.05 065	9.87 334	12	**20**
41	9.82 283	14	9.94 961	25	10.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	10.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	10.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	10.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	10.04 938	9.87 277	11	**15**
46	9.82 354	14	9.95 088	25	10.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	10.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	10.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	10.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	10.04 810	9.87 221	12	**10**
51	9.82 424	15	9.95 215	25	10.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	10.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	10.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	10.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	10.04 683	9.87 164	11	**5**
56	9.82 495	14	9.95 342	26	10.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	10.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	10.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	10.04 582	9.87 119	12	1
60	9.82 551		9.95 444		10.04 556	9.87 107		**0**
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	12	11
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

From the top:

For **41°+** or **221°+**, read as printed; for **131°+** or **311°+**, read co-function.

From the bottom:

For **48°+** or **228°+**, read as printed; for **138°+** or **318°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.82 551	14	9.95 444	25	10.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	10.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	10.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	10.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	10.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	10.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	10.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	10.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	10.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	10.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	10.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	10.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	10.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	10.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	10.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	10.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	10.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	10.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	10.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	10.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	10.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	10.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	10.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	10.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	10.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	10.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	10.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	10.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	10.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	10.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	10.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	10.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	10.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	10.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	10.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	10.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	10.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	10.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	10.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	10.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	10.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	10.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	10.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	10.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	10.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	10.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	10.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	10.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	10.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	10.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	10.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	10.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	10.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	10.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	10.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	10.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	10.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	10.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	10.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	10.03 060	9.86 425	12	1
60	9.83 378		9.96 966		10.03 034	9.86 413		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

From the top:

For 42°+ or 222°+, read as printed; for 132°+ or 312°+, read co-function.

From the bottom:

For 47°+ or 227°+, read as printed; for 137°+ or 317°+, read co-function.

Prop. Pts.

47° — Logarithms of Trigonometric Functions

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	
0	9.83 378	14	9.96 966	25	10.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	10.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	10.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	10.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	10.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	10.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	10.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	10.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	10.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	10.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	10.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	10.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	10.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	10.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	10.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	10.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	10.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	10.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	10.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	10.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	10.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	10.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	10.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	10.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	10.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	10.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	10.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	10.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	10.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	10.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	10.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	10.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	10.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	10.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	10.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	10.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	10.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	10.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	10.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	10.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	10.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	10.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	10.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	10.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	10.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	10.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	10.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	10.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	10.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	10.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	10.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	10.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	10.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	10.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	10.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	10.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	10.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	10.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	10.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	10.01 542	9.85 706	13	1
60	9.84 177		9.98 484		10.01 516	9.85 693		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

From the top:

For **43°+** or **223°+**, read as printed; for **133°+** or **313°+**, read co-function.

From the bottom:

For **46°+** or **226°+**, read as printed; for **136°+** or **316°+**, read co-function.

Prop. Pts.

′	L Sin	d	L Tan	cd	L Ctn	L Cos	d	′
0	9.84 177	13	9.98 484	25	10.01 516	9.85 693	12	60
1	9.84 190	13	9.98 509	25	10.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	10.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	10.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	10.01 415	9.85 645	13	56
5	9.84 242	13	9.98 610	25	10.01 390	9.85 632	12	55
6	9.84 255	14	9.98 635	26	10.01 365	9.85 620	12	54
7	9.84 269	13	9.98 661	25	10.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	10.01 314	9.85 596	13	52
9	9.84 295	13	9.98 711	26	10.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	10.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	10.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	10.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	10.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	10.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	10.01 137	9.85 510	13	45
16	9.84 385	13	9.98 888	25	10.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	10.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	25	10.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	10.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	10.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015	25	10.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	10.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	10.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	26	10.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	10.00 884	9.85 386	12	35
26	9.84 515	13	9.99 141	25	10.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	10.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	10.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	10.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	10.00 758	9.85 324	12	30
31	9.84 579	13	9.99 267	26	10.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	10.00 707	9.85 299	12	28
33	9.84 605	13	9.99 318	25	10.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	10.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	10.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	25	10.00 606	9.85 250	13	24
37	9.84 656	13	9.99 419	25	10.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	10.00 556	9.85 225	13	22
39	9.84 682	12	9.99 469	26	10.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495	25	10.00 505	9.85 200	13	20
41	9.84 707	13	9.99 520	25	10.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	10.00 455	9.85 175	13	18
43	9.84 733	12	9.99 570	26	10.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	10.00 404	9.85 150	13	16
45	9.84 758	13	9.99 621	25	10.00 379	9.85 137	12	15
46	9.84 771	13	9.99 646	26	10.00 354	9.85 125	13	14
47	9.84 784	12	9.99 672	25	10.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	10.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	10.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	10.00 253	9.85 074	12	10
51	9.84 835	12	9.99 773	25	10.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	10.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	10.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	10.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	10.00 126	9.85 012	13	5
56	9.84 898	13	9.99 899	25	10.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	10.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	10.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975	25	10.00 025	9.84 961	12	1
60	9.84 949		10.0000		10.00 000	9.84 949		0
	L Cos	d	L Ctn	cd	L Tan	L Sin	d	′

Prop. Pts.

	26	25
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

From the top:

For 44°+ or 224°+, read as printed; for 134°+ or 314°+, read co-function.

From the bottom:

For 45°+ or 225°+, read as printed; for 135°+ or 315°+, read co-function.

Prop. Pts.

ANSWERS TO ODD NUMBERED EXERCISES AND PROBLEMS

The answers to even numbered exercises are omitted for reasons stated in the preface. Those of some odd numbered exercises are also intentionally omitted.

CHAPTER I

Article 2. Page 2

1. $4/5, 3/5, 4/3$ **3.** $24/25, 7/25, 24/7; 7/25, 24/25, 7/24$
5. $\sqrt{3}/2, 1/2, \sqrt{3}$ **7.** $1/2, \sqrt{3}/2, \sqrt{3}/3$ **9.** $0.70, 0.72, 0.97$

Article 3. Page 3

3. $41/9, 41/40, 40/9$

Article 4. Page 4

3. $\sin A = 5/13, \quad \cos A = 12/13, \quad \tan A = 5/12,$
$\cot A = 12/5, \quad \sec A = 13/12, \quad \csc A = 13/5$

Article 7. Pages 6, 7

1. 500 ft. **7.** $80\sqrt{3}/3$ ft. **9.** $45°, 30°$
15. 6 **17.** 1 **19.** 347 **21.** $\frac{5}{2}(1 + \sqrt{3})$

Article 8. Page 9

1. $\cos A = 3/5, \tan A = 4/3, \cot A = 3/4, \sec A = 5/3, \csc A = 5/4$
3. $\sin A = 7/25, \cos A = 24/25, \tan A = 7/24, \sec A = 25/24, \csc A = 25/7$
5. $\sin A = 5/13, \cos A = 12/13, \cot A = 12/5, \sec A = 13/12, \csc A = 13/5$
7. $\sin A = 8/17, \tan A = 8/15, \cot A = 15/8, \sec A = 17/15, \csc A = 17/8$
9. $\sin A = 1/2, \cos A = \sqrt{3}/2, \tan A = \sqrt{3}/3, \sec A = 2\sqrt{3}/3, \csc A = 2$
11. $\sin A = \sqrt{3}/2, \cos A = 1/2, \tan A = \sqrt{3}, \sec A = 2, \csc A = 2\sqrt{3}/3$
13. $\cos A = \sqrt{3}/2, \tan A = \sqrt{3}/3, \cot A = \sqrt{3}, \sec A = 2\sqrt{3}/3, \csc A = 2$
15. $\sin A = 40/41, \cos A = 9/41, \tan A = 40/9, \cot A = 9/40, \sec A = 41/9$
17. $\sin A = 4/5, \cos A = 3/5, \tan A = 4/3, \sec A = 5/3, \csc A = 5/4$
19. 12 **21.** 16 **23.** 200 **25.** $a = 60, b = 80$

Article 9. Page 10

1. $\cos 60°$ **3.** $\cot 20°$ **5.** $\csc 78° 35'$ **7.** $\sin 22° 30'$
9. $\sin 75° = \cos 15°, \quad \cos 75° = \sin 15°, \quad \tan 75° = \cot 15°,$
$\cot 75° = \tan 15°, \quad \sec 75° = \csc 15°, \quad \csc 75° = \sec 15°$
11. $15°$ **13.** $15°$ **15.** $15°$

Article 10. Page 12

1. $\sin^2 x$ **3.** 1

13. $\sin A = \dfrac{\tan A\sqrt{1 + \tan^2 A}}{1 + \tan^2 A}, \quad \cos A = \dfrac{\sqrt{1 + \tan^2 A}}{1 + \tan^2 A},$

$\cot A = \dfrac{1}{\tan A}, \quad \sec A = \sqrt{1 + \tan^2 A}, \quad \csc A = \dfrac{\sqrt{1 + \tan^2 A}}{\tan A}$

CHAPTER II

Article 12. Pages 14, 15

1. 0.267 **3.** 0.924 **5.** 0.754 **7.** 5.976 **9.** 0.719
11. 0.415 **13.** 4.165 **15.** 0.566 **17.** 14° **19.** 54°
21. 23° **23.** 43° 30′ **25.** 16° **27.** 35° **29.** 72°
31. 66° **33.** 70° **35.** 66° **37.** 22° **39.** 2°

Article 13. Pages 18, 19

1. 295 ft. **3.** 32.1 ft. **5.** 119 yds. **7.** 456 ft.
9. 94.5 ft. **11.** 767 ft. **13.** 52° 30′ **15.** 12°, 53°

Article 14. Pages 20, 21

1. 30.9 ft. **3.** 20° **5.** 79° 30′ **7.** 16° 30′ to 35° **9.** 27°

Article 15. Pages 22–24

1. N 38° E **3.** 29° **5.** 96.6 ft. **7.** 314 yds. **9.** 73.3 rds. **11.** 109 ft.
13. 34.3 ft., 41° **15.** 65 ft., 69° 30′ **19.** 12800 ft., 2350 ft. **21.** 146 ft., 620 ft.

Article 17. Pages 26, 27

1. $B = 58°\,30'$, $b = 44.1$, $c = 51.7$
3. $B = 43°\,30'$, $a = 25.8$, $c = 35.6$
5. $A = 41°\,30'$, $B = 48°\,30'$, $c = 24.1$
7. $A = 44°$, $B = 46°$, $a = 69.4$
9. $B = 66°\,30'$, $a = 34.8$, $c = 87.2$
11. 9.49 ft.
13. 145°
15. The first course.
17. 71.3 ft.
19. 77.0 in.

Article 19. Pages 29, 30

1. 78.296, 0.014, 1.112, 0.001
3. 78.2961, 0.0138, 1.1125, 0.0007
78.30, 0.01384, 1.112, 0.0007220
5. 810, 810, 810
7. 15, 15.3, 15.28
11. 38.1
13. 69.5
15. 14.3, 0.75, 4444.
17. 18°

CHAPTER III

Article 23. Page 32

1. II, I, III, II, III, IV, III, I
3. 30°, 390°; 30°, − 330°; 390°, − 330°
5. 439°, 799°, − 281°, − 641°
7. − 1080°
9. II **11.** IV
13. 9000 rev. per hr.

Article 25. Page 34

1. $-12/13,\ 12/5,\ -13/5$
3. $-12/13,\ -12/5,\ 13/5$
5. $-2/\sqrt{4+b^2},\ -b/2,\ \sqrt{4+b^2}/b$
7. $-4/5,\ 4/3,\ -5/3;\ -3/5,\ 3/4,\ -5/4$

Article 26. Pages 36–38

In this section the answer to each question follows the order sin, cos, tan, csc, sec, cot.

5. $-\sqrt{2}/2,\ -\sqrt{2}/2,\ 1,\ -\sqrt{2},\ -\sqrt{2},\ 1$
7. $-\sqrt{2}/2,\ \sqrt{2}/2,\ -1,\ -\sqrt{2},\ \sqrt{2},\ -1$; same as for 315°

9. $-\frac{1}{2}$, $-\sqrt{3}/2$, $\sqrt{3}/3$, -2, $-2\sqrt{3}/3$, $\sqrt{3}$; same as for 210°

11. For 300° and $-60°$: $-\sqrt{3}/2$, $1/2$, $-\sqrt{3}$, $-2\sqrt{3}/3$, 2, $-\sqrt{3}/3$
For 330° and $-30°$: $-1/2$, $\sqrt{3}/2$, $-\sqrt{3}/3$, -2, $2\sqrt{3}/3$, $-\sqrt{3}$

13. $3\sqrt{3}$ 15. $-3/4$ 17. $-9/4$

21. $3\sqrt{5}/7$, $2/7$, $3\sqrt{5}/2$, $7\sqrt{5}/15$, $7/2$, $2\sqrt{5}/15$;
$-3\sqrt{5}/7$, $2/7$, $-3\sqrt{5}/2$, $-7\sqrt{5}/15$, $7/2$, $-2\sqrt{5}/15$

23. $1/3$, $2\sqrt{2}/3$, $\sqrt{2}/4$, 3, $3\sqrt{2}/4$, $2\sqrt{2}$;
$1/3$, $-2\sqrt{2}/3$, $-\sqrt{2}/4$, 3, $-3\sqrt{2}/4$, $-2\sqrt{2}$

25. $1/5$, $2\sqrt{6}/5$, $\sqrt{6}/12$, 5, $5\sqrt{6}/12$, $2\sqrt{6}$;
$-1/5$, $-2\sqrt{6}/5$, $\sqrt{6}/12$, -5, $-5\sqrt{6}/12$, $2\sqrt{6}$

27. $\sqrt{1-m^2}, m, \dfrac{\sqrt{1-m^2}}{m}, \dfrac{1}{\sqrt{1-m^2}}, \dfrac{1}{m}, \dfrac{m}{\sqrt{1-m^2}}$;
$-\sqrt{1-m^2}, m, \dfrac{-\sqrt{1-m^2}}{m}, -\dfrac{1}{\sqrt{1-m^2}}, \dfrac{1}{m}, -\dfrac{m}{\sqrt{1-m^2}}$

29. $\dfrac{l}{m}, \dfrac{\sqrt{m^2-l^2}}{m}, \dfrac{l}{\sqrt{m^2-l^2}}, \dfrac{m}{l}, \dfrac{m}{\sqrt{m^2-l^2}}, \dfrac{\sqrt{m^2-l^2}}{l}$;
$\dfrac{l}{m}, \dfrac{-\sqrt{m^2-l^2}}{m}, -\dfrac{l}{\sqrt{m^2-l^2}}, \dfrac{m}{l}, \dfrac{-m}{\sqrt{m^2-l^2}}, \dfrac{-\sqrt{m^2-l^2}}{l}$

31. $\dfrac{l}{m}, \dfrac{-\sqrt{m^2-l^2}}{m}, \dfrac{l}{-\sqrt{m^2-l^2}}, \dfrac{m}{l}, \dfrac{-m}{\sqrt{m^2-l^2}}, \dfrac{-\sqrt{m^2-l^2}}{l}$;
$\dfrac{l}{m}, \dfrac{\sqrt{m^2-l^2}}{m}, \dfrac{l}{\sqrt{m^2-l^2}}, \dfrac{m}{l}, \dfrac{m}{\sqrt{m^2-l^2}}, \dfrac{\sqrt{m^2-l^2}}{l}$

33. $\dfrac{l}{\sqrt{l^2+m^2}}, \dfrac{m}{\sqrt{l^2+m^2}}, \dfrac{l}{m}, \dfrac{\sqrt{l^2+m^2}}{l}, \dfrac{\sqrt{l^2+m^2}}{m}, \dfrac{m}{l}$;
$\dfrac{-l}{\sqrt{l^2+m^2}}, \dfrac{-m}{\sqrt{l^2+m^2}}, \dfrac{l}{m}, \dfrac{-\sqrt{l^2+m^2}}{l}, \dfrac{-\sqrt{l^2+m^2}}{m}, \dfrac{m}{l}$

37. $-1/2$, $-\sqrt{3}/2$, $\sqrt{3}/3$, -2, $-2\sqrt{3}/3$, $\sqrt{3}$

39. 0.6, -0.8, -0.75, 5/3, $-5/4$, $-4/3$

Article 28. Page 40

3. 0 5. 0 7. 1, 0, ∞, 1, ∞, 0; -1, 0, ∞, -1, ∞, 0

9. 0, -1, 0, ∞, -1, ∞

Article 30. Page 42

3. cos 130° 5. $-$ tan 410° 7. cos 1511° 9. $-$ sin 75°
11. tan 45° 13. $-$ sec 380° 15. $-$ cos 59° 17. $-$ cot 948°
19. -0.891 21. -2 23. -2 27. 1.428
33. $-\sqrt{2}/2$, $-\sqrt{2}/2$, 1 35. $-\sqrt{3}/2$, 1/2, $-\sqrt{3}$ 37. $\sqrt{3}/2$, $-1/2$, $-\sqrt{3}$

Article 31. Page 43

1. sin 9° 3. cos 76° 5. $-$ tan 58°

Article 32. Page 45

1. sin 25°, $-$ cos 20°, $-$ tan 59° 3. $-$ sin 43°, $-$ cos 83°, tan 27°
5. $-$ sin 33°, $-$ cos 10°, tan 51°

Article 33. Pages 45, 46

3. sin 15° 7. sin 45° 30′ 11. − csc 5° 15. 0.983
5. − sec 31° 9. cot 10° 30′ 20″ 13. − 0.814

Miscellaneous Exercises. Pages 46, 47

1. − 170° 5. II, I 9. − 3/5, − 4/5, − 5/3
3. − 113640° 7. ± 5/13, ± 12/5, ± 13/5 11. − 4 − $\sqrt{3}/2$

13. 1/4 15. − 2 17. $l, -\sqrt{1-l^2}, \frac{-l}{\sqrt{1-l^2}}, \frac{1}{l}, \frac{-1}{\sqrt{1-l^2}}, \frac{-\sqrt{1-l^2}}{l}$

19. − 1/2 21. 1200° 25. $\sqrt{1-a^2}, -a, \frac{-\sqrt{1-a^2}}{a}, \frac{1}{\sqrt{1-a^2}}, -\frac{1}{a}, \frac{-a}{\sqrt{1-a^2}}$

29. 210°, 330° 31. 120°, 300° 33. 150°, 210° 37. 1 39. $-(a+b)\tan x$

CHAPTER V

Article 40. Page 58

1. 1 3. 0, ± 180°, ± 360°, ⋯ 5. 2

Article 40. Page 59

3. ± 90°, ± 270°, ⋯

Article 40. Page 60

1. No 3. No; ± 90°, ± 270°, ⋯

Article 41. Page 60

3. 1 5. $\pm n \cdot 180°$, where n is zero or a positive integer.

CHAPTER VI

Article 43. Page 64

1. 39.56 3. 112.9 5. 37° 8′

Article 44. Pages 65, 66

1. 786.7 ft. 3. 933.0 ft., 701.4 ft
5. 18.28 mi. 9. 43.97, 17.36 11. 76.90

Article 45. Pages 68, 69

1. 17.65 3. 118.5 5. 61° 17′ 11. $50\sqrt{10}$

Article 46. Pages 70, 71

1. 48° 52′, 58° 36′, 72° 32′ 5. 524.0
3. 32.36 ft., 67° 34′ 7. 60.85
9. $9\sqrt{34+15\sqrt{2}} = 66.87$ mi. ($\sqrt{2} = 1.414$)
11. 65.79 lbs., 51.62 lbs. 17. 35.45 ft. 19. 23.19 ft.

Article 47. Page 73

1. 363.1 3. 573.2 5. 6248 7. 93.90 9. 5700
13. 211.5 sq. ft. 17. 4.173, 5.033, 32.57 19. 490.9

CHAPTER VII

Article 51. Pages 78, 79

7. 56/33, 56/65, 33/65, 16/65 **9.** $2(1-\sqrt{10})/9$, $2(1+\sqrt{10})/9$
23. 4/5, 3/5, 4/3 **25.** 1/2 **27.** $-\sqrt{2}/10$, $-1/7$
29. $\sqrt{3}$ **33.** $-1/2$ **35.** $\sqrt{3}/3$

Article 52. Page 80

5. 24/25 **7.** $-120/119$

Article 53. Pages 81, 82

1. 24/25, $-7/25$ **3.** 24/25, 24/7 **5.** $-7/25$ **7.** 1
9. $-\sqrt{2}/2$ **11.** $(\sqrt{6}+\sqrt{2})/4$ **13.** $\frac{1}{2}\sqrt{2-\sqrt{2}}$ **15.** $\sqrt{2}-1$
27. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **31.** $\sqrt{3}/2$, 1/2 **33.** $(1-\cos 6x)/2$
35. $(1-\cos 40°)/2$ **51.** $\sin 2x$ **53.** $\tan 2x$ **55.** $\tan(x/2)$

Article 55. Pages 84–86

1. $\sin 4A + \sin 2A$ **3.** $\sin 8x - \sin 2x$ **5.** $\cos 4\theta - \cos 10\theta$
7. $(\cos 2A + \cos 2B)/2$ **9.** $\sqrt{6}/2$ **11.** $\sqrt{2}/2$
13. $\sqrt{2}\sin 3°$ **15.** $\sqrt{3}\cos 10°$ **17.** $\sin 5°$
19. $\sin 10°$ **21.** $-\sqrt{2}/2$ **23.** $2\sin 3\theta\cos 2\theta$
25. $2\cos 3x\sin 2x$ **27.** $2\sin A\cos B$ **29.** $2\cos 3x\sin 2x$

CHAPTER VIII

Article 63. Page 92

1. 1.82607 **3.** 3.51268 **5.** 3.30103 **7.** 9.74819 − 10
9. 1.25200 **11.** 9.89587 − 10 **13.** 0.82667 **15.** 2.09311

Article 64. Page 93

1. 136.70 **3.** 3012.0 **5.** 0.60984 **7.** 371.53 **9.** 6.0263 **11.** 100.75

Article 65. Pages 94–96

1. 665.15 **3.** 11.446 **5.** 11.559 **7.** 0.14142 **9.** − 0.91656
11. 0.052781 **13.** 2.5418 **15.** 0.057530 **17.** 111.92 **19.** 0.0000014444
21. − 118.08 **23.** 0.00066185 **25.** 14.782 **27.** 99022. **29.** 10.300 in.
31. 48.057 yds. **33.** 1004700 **35.** 245.94 **37.** 99.498 cm. **39.** 156.25 cm.

Article 68. Pages 98, 99

1. 9.36660 **3.** 0.05497 **5.** 9.99183 **7.** 9.89744 **9.** 9.92947
11. 9.89687 **13.** 0.57800 **15.** 1.01326 **17.** 9.94423 **19.** 0.03759
21. 9.91206 **23.** 0.43295 **25.** 52° 17′ **27.** 25° 26′ **29.** 39° 8.8′
31. 80° 37.2′ **33.** 16° 25.1′ **35.** 84° 15.7′ **37.** 31° 27.1′ **39.** 45°
41. 17° 5.5′, 342° 54.5′ **43.** 65° 52.2′, 245° 52.2′
45. 81° 6.8′, 278° 53.2′ **47.** 79° 59.4′, 259° 59.4′
49. 3135.5 **51.** 3221.6 **53.** 242.24

Article 70. Page 100

1. 19° 21′ 24″ **3.** 50° 43′ 42″ **5.** 41° 41′ 48″ **7.** 17° 19.4′ **9.** 46° 41.3′
11. 39° 40.7′ **15.** 9.82249 **17.** 0.40336 **19.** 9.16151 **21.** 9.99363
23. 36° 52′ 32″ **25.** 58° 10′ 41″ **27.** 52° 40′ 56″ **29.** 17° 0′ 35″

CHAPTER IX

Article 74. Pages 103, 104

1. $B = 43° 33'$, $a = 1651.7$, $b = 1570.2$
3. $A = 40° 32.5'$, $B = 49° 27.5'$, $b = 151.99$
5. $A = 72° 41.4'$, $B = 17° 18.6'$, $a = 938.18$
7. $A = 24° 44.4'$, $a = 849.54$, $c = 2029.9$
9. 70.134 ft. **13.** 100.01 ft. **15.** 340.83 ft.

Article 76. Pages 105, 106

1. $C = 101° 57'$, $a = 191.22$, $c = 1974.5$
3. $C = 69° 13.7'$, $b = 4627.4$, $c = 4989.9$
5. $A = 57° 23.2'$, $a = 754.97$, $c = 832.53$ **7.** 368.78 yds., 177.86 yds.

Article 77. Pages 109, 110

1. $A = 92° 43.5'$, $C = 47° 16.5'$, $a = 543.89$
$A' = 7° 16.5'$, $C' = 132° 43.5'$, $a' = 68.953$
3. $B = 26° 19'$, $C = 26° 8'$, $b = 100.65$
5. $B = 42° 38.1'$, $C = 82° 21.9'$, $c = 619.86$ **7.** No solution
9. $B = 66° 39.6'$, $C = 19° 2.8'$, $c = 142.28$
11. $A = 156° 40.8'$, $C = 6° 24.2'$, $a = 13.918$ **13.** 13585 ft.; 2614.9 ft.

Article 79. Page 112

1. $A = 41° 57.0'$, $B = 38° 3.0'$, $c = 447.86$
3. $A = 77° 25.0'$, $C = 50° 34.6'$, $b = 403.75$
5. $B = 47° 48.2'$, $C = 64° 44.6'$, $a = 1014.7$ **7.** 760.58 dynes

Article 81. Pages 115, 116

1. $A = 82° 49.2'$, $B = 55° 46.2'$, $C = 41° 24.6'$
3. $A = 82° 21.6'$, $B = 54° 56.4'$, $C = 42° 42.2'$
5. $A = 44° 24.8'$, $B = 57° 7.2'$, $C = 78° 27.8'$
7. S 45° 20.4′ E **9.** 119° 12.0′

Article 82. Page 116

1. 748.03 **3.** 187.83 **5.** 66837 **7.** 1428.7 ft.

Miscellaneous Exercises. Pages 119–123

1. $A = 13° 23'$, $a = 58.270$, $c = 251.75$
3. $A = 57° 53.5'$, $B = 32° 6.5'$, $b = 42.375$
5. $A = 40° 12.6'$, $a = 613.34$, $b = 725.52$
7. $A = 37° 13'$, $a = 3.4670$, $b = 5.0597$
9. $C = 14° 38.9'$, $b = 255.21$, $c = 74.343$

11. $A = 92° 36.2'$, $B = 48° 38.4'$, $a = 312.77$
$A' = 9° 53.0'$, $B' = 131° 21.6'$, $a' = 53.740$
13. No solution
15. $B = 35° 5.2'$, $C = 86° 13.0'$, $a = 60.676$
17. $A = 82° 49.1'$, $B = 55° 46.3'$, $c = 4000.0$
19. $A = 33° 34.8'$, $B = 18° 21.4'$, $C = 128° 3.8'$
21. $A = 17° 2.6'$, $B = 17° 58.2'$, $C = 144° 59.2'$
23. 70.003 25. 34° 5.2′, 85° 6.8′, 60° 48′, 1598.0 27. 13684
29. 32.937; 76.180; 77.353 31. 9.7704 ft. 33. 476.58; 426.09; 344.18
35. 48° 11.4′, 58° 24.6′, 73° 23.8′ 37. 78.750 ft.; 175.00 ft.
39. 1271.1 sq. ft. 41. 355.18 ft.
53. $30(1 + \sqrt{3})$ ft. 57. 47° 52′, or 132° 8′
59. 656.73 ft. 61. 553.14 yds., 405.05 yds.
63. $b = 29.67$, $a = 999.6$, $B = 1° 42'$

CHAPTER X

Article 86. Pages 125, 126

1. $2\pi/3$ 3. $5\pi/6$ 5. $5\pi/4$ 7. $3\pi/2$ 9. $11\pi/6$ 11. $\pi/3$
13. $-4\pi/3$ 15. $-\pi/3$ 17. 90° 19. 60° 21. 270° 23. 120°
25. 150° 27. −180° 29. 57.296° 31. 143.24° 33. 229.18° 35. −18°
37. $-\pi/2$, −90° 39. $5\pi/12$ 41. 2.0944 43. 0.7389 45. 1.2116 47. 1.0807
49. 1 51. 1 53. 1/2 55. −1 57. −1 59. 0

Article 88. Pages 127, 128

1. 12 in. 3. 2π in. 5. 4140.8 ft. 7. 3374 mi. 9. 70 ft.
13. 7500, 7854, 5236 sq. in. 15. 4473 sq. ft.

CHAPTER XI

Article 89. Page 130

3. *a.* $\cos y = x$ *b.* $y = \text{arc}\tan x$ *c.* $z = \text{arc}\sec x$
d. $\csc y = x$ *e.* $\cot z = x$ *f.* $\sin u = v$
5. $y = \text{arc}\cos(-1/2)$ 7. $y = \text{arc}\sec(-3)$ 9. $\theta = \text{arc}\cot 2$
11. 45°, $\pi/4$ 13. 45°, $\pi/4$ 15. 30°, $\pi/6$ 17. 60°, $\pi/3$ 19. 60°, $\pi/3$
21. 0°, 0 23. x 25. x 27. x

Article 92. Page 132

1. 30°, $\pi/6$ 3. 120°, $2\pi/3$ 5. 150°, $5\pi/6$ 7. 45°, $\pi/4$
9. −45°, $-\pi/4$ 11. 0°, 0 13. 0°, 0 15. −30°, $-\pi/6$
17. −60°, $-\pi/3$ 19. 90°, $\pi/2$ 21. 60°, $\pi/3$ 23. 180°, π
25. 135°, $3\pi/4$ 27. −90°, $-\pi/2$ 29. 17° 5′ 31. 23° 42′

Article 93. Page 134

1. $\sqrt{3}$ 3. $-\sqrt{2}/2$ 5. $\sqrt{2}$ 7. $-12/13$ 9. -1 11. 3/5
13. 12/13 15. $\sqrt{3}/2$ 17. $-2/3$ 19. $2x\sqrt{1-x^2}$ 21. $2x/(1-x^2)$
23. $y\sqrt{1-x^2} - x\sqrt{1-y^2}$ 25. $x\sqrt{1-y^2} - y\sqrt{1-x^2}$
27. $(y-x)/(xy+1)$. 29. $(1-x)/2$

CHAPTER XII

Article 96. Page 139

1. 135°, 315° **3.** $\pi/3, 2\pi/3, 4\pi/3, 5\pi/3$ radians **5.** 28° 50′, 151° 10′
7. 40° **9.** $90° + n \cdot 180° = \pi/2 + n \cdot \pi$; $45° + n \cdot 180° = \pi/4 + n \cdot \pi$

Article 97. Page 142

1. 30°, 150°, 210°, 330° **3.** 30°, 150°, 270° **5.** 45°, 225°, 153° 26′, 333° 26′
7. $0, \pi, \pi/6, 5\pi/6$ **9.** 36° 52′, − 90° **11.** $0, \pi/4, \pm \pi/2$
13. $63° 26' + n \cdot 180°$; $1.107 + n \cdot \pi$ radians
15. $22\frac{1}{2}°[1 - (-1)^n] - n \cdot 90° = \pi/8[1 - (-1)^n] - n\pi/2$ radians
17. $r = 3, \theta = + \pi/3, 5\pi/3$ **19.** $r = 5, \theta = \text{arc} \sin 3/5 = 36° 52'$
21. $x = 1.17$ radians, $y = 2.34$

CHAPTER XIII

Article 102. Page 149

1. $a = 49° 15.6'$, $b = 76° 50.5'$, $c = 81° 27.4'$
3. $A = 66° 12.2'$, $c = 107° 5.0'$, $b = 127° 17.7'$

Article 102. Page 151

1. $A = 39° 20.7'$, $B = 85° 21.3'$, $c = 84° 18.7'$
3. $a = 55° 47.7'$, $b = 65° 29.0'$, $c = 76° 30.6'$

Article 103. Page 152

1. $a = 54° 59.8'$, $B = 45° 41.4'$, $A = 65° 46'$
3. $A = 65° 49.9'$, $c = 63° 10.1'$, $b = 38° 59.2'$
5. $A = 71° 36.0'$, $B = 30° 51.3'$, $c = 56° 9.6'$
7. $A = 71° 36.8'$, $a = 71° 32.2'$, $b = 95° 22.5'$
9. $a = 49° 15.6'$, $b = 76° 50.5'$, $c = 81° 27.3'$
11. $a = 34° 34.2'$, $B = 81° 24.4'$, $c = 77° 46.0'$
13. $a = 23° 39.1'$, $b = 46° 54.8'$, $c = 51° 15.9'$
15. $a = 20° 46.0'$, $c = 36° 21.6'$, $B = 58° 59.7'$
17. $A = 46° 50.2'$, $b = 71° 41.5'$, $C = 106° 24.9'$
19. $A = 90°$, $B = 90°$, $C = 90°$
21. $a = 73° 17.5'$, $C = 96° 13.6'$, $b = 70° 8.6'$
23. 55° 44.1′

CHAPTER XIV

Article 112. Page 162

1. $A = 115° 3.4'$, $B = 120° 11.2'$, $C = 127° 11.4'$
3. $A = 103° 52'$, $B = 123° 55.6'$, $C = 82° 46.2'$
5. $a = 149° 24.2'$, $b = 129° 48'$, $c = 67° 18.8'$
7. $a = 99° 1.2'$, $b = 73° 53.4'$, $c = 97° 53.2'$
9. $a = 95° 37.8'$, $b = 41° 52.4'$, $C = 110° 48.4'$
11. $a = 113° 2.9'$, $c = 74° 54.3'$, $B = 75° 1'$
13. $A = 131° 28.8'$, $C = 29° 32.8'$, $b = 72° 40.6'$
15. $A = 69° 47.7'$ or $110° 12.3'$
$C = 94° 39.2'$ or $148° 4.8'$
$c = 115° 45.2'$ or $151° 27.4'$

17. No solution

19. $A = 164° 43.8'$ or $119° 18.6'$

$a = 162° 37.6'$ or $81° 18.6'$

$c = 124° 40.6'$ or $55° 19.4'$

25. $a = 15.7$ in., $b = 10.7$ in., $c = 12.7$ in.

Article 113. Page 163

1. $51° 47'$, 0.9037 radian

3. 6.283

5. 4645

7. 201.5 or 365.3

CHAPTER XV

Article 116. Page 166

1. 2567 miles, N 69° 44′ E

3. 5663 miles

5. 14 hours 51 minutes

Article 118. Page 168

1. 2 hours 42.9 minutes

3. 43.3 minutes after 4 P.M.

5. 14.5 minutes after 5 A.M.

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